STAT 501: Introduction to Probabilities

Homework and Midterm Solutions

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Contents

1 Homework 1

Problem 1.9.6

Suppose $a_n > 0$, $b_n > 1$, and

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 1$$

Define

$$A_n := \{x : a_n \leqslant x < b_n\}$$

Find

$$\limsup_{n\to\infty} A_n \qquad \text{and} \qquad \liminf_{n\to\infty} A_n$$

By inspection, we have the following claim

$$\begin{aligned}\limsup_{n\to\infty} A_n &= \cap_{n\geqslant 1} \cup_{k\geqslant n} A_k \\ &= \cap_{n\geqslant 1} \cup_{k\geqslant n} \{x : a_k \leqslant x < b_k\} \\ &= (0,1]\end{aligned}$$

Proof $\limsup_{n\to\infty} A_n$

Show $\limsup_{n\to\infty} A_n \subseteq (0,1]$

- Let $x \in \limsup_{n\to\infty} A_n = \cap_{n\geqslant 1} \cup_{k\geqslant n} \{y : a_k \leqslant y < b_k\}$.
- Here we prove by contradiction. Suppose $x \notin (0,1]$. Then we have two cases.
  1) Case 1: $x \leqslant 0$.
     - Since $\lim_{n\to\infty} a_n = 0$ with $a_n > 0$, then for all $k \geqslant 1$, $x < a_k$.
     - i.e. for all $k \geqslant 1$, $x \notin A_k$.
     - i.e. for any $n \geqslant 1$, $x \notin \bigcup_{k\geqslant n} A_k$, since no member of that union contains x.
     - Hence $x \notin \cap_{n\geqslant 1} \cup_{k\geqslant n} A_k = \limsup_{n\to\infty} A_n$. A contradiction.
  2) Case 2: $x > 1$.
     - Since $\lim_{n\to\infty} b_n = 1$ with $b_n > 1$, there exists some $n_0 \in \mathbb{Z}_{\geqslant 1}$ such that for all $k \geqslant n_0$, $x \geqslant b_k$.
     - Equivalently, there exists some $n_0 \in \mathbb{Z}_{\geqslant 1}$ such that for all $k \geqslant n_0$, $x \notin A_k$.
     - Equivalently, there exists some $n_0 \in \mathbb{Z}_{\geqslant 1}$ such that $x \notin \cup_{k\geqslant n_0} A_k$
     - Hence $x \notin \cap_{n\geqslant 1} \cup_{k\geqslant n} A_k = \limsup_{n\to\infty} A_n$
- Hence, by proof of contradiction, we have $x \in (0,1]$

Show $\limsup_{n\to\infty} A_n \supseteq (0,1]$

- Let $x \in (0,1]$.
- Consider two cases
  1) Case 1: interior case, $x \in (0,1)$.
     - Since $a_n \to 0$ and $b_n \to 1$, there exists n_0 such that for every $k \geqslant n_0$, $x \in \{y : a_k \leqslant y < b_k\} = A_k$.
     - Hence, $x \in \cap_{k\geqslant n} A_k \subseteq \cup_{k\geqslant n} A_k$ for any $n \geqslant 1$.
     - Hence, $x \in \cap_{n\geqslant 1} \cup_{k\geqslant n} A_k = \limsup_{n\to\infty} A_k$.
  2) Case 2: boundary case, $x = 1$
     - Since $\lim_{n\to\infty} a_n = 0$, there exists $n_0 \in \mathbb{Z}_{\geqslant 1}$ such that $a_k < 1$ for all $k \geqslant n_0$; and $b_k > 1$ for every $k \geqslant 1$. Hence for all $k \geqslant n_0$, $a_k \leqslant 1 < b_k$,
     - i.e. $1 \in A_k$ for every $k \geqslant n_0$
     - Hence for any $n \geqslant 1$, picking any $k \geqslant \max(n, n_0)$ gives $1 \in A_k$, so $1 \in \bigcup_{k\geqslant n} A_k$.
     - Hence, $1 \in \cap_{n\geqslant 1} \cup_{k\geqslant n} A_k$
- Hence, we showed $x \in \limsup_{n\to\infty} A_n$.

Proof $\liminf_{n\to\infty} A_n$

$$\begin{aligned}
\liminf_{n\to\infty} A_n &= \cup_{n\geqslant 1} \cap_{k\geqslant n} A_k \\
&= \cup_{n\geqslant 1} \cap_{k\geqslant n} \{x : a_k \leqslant x < b_k\} \\
&= (0,1]
\end{aligned}$$

Show $\liminf_{n\to\infty} A_n \subseteq (0,1]$

- Let $x \in \liminf_{n\to\infty} A_n = \cup_{n\geqslant 1} \cap_{k\geqslant n} \{y : a_k \leqslant y < b_k\}$.
- Assume by contradiction that $x \notin (0,1]$. Two cases
  1) Suppose $x \leqslant 0$
     - Since $\lim_{n\to\infty} a_n = 0$ with $a_n > 0$, then for all $k \geqslant 1$, $x < a_k$.
     - i.e. for all $k \geqslant 1$, $x \notin A_k$.
     - i.e. $x \notin \cap_{k\geqslant n} A_k$ for any $n \geqslant 1$.
     - Hence $x \notin \bigcup_{n\geqslant 1} \bigcap_{k\geqslant n} A_k = \liminf_{n\to\infty} A_n$. A contradiction.
  2) Suppose $x > 1$
     - Since $\lim_{n\to\infty} b_n = 1$ with $b_n > 1$. Pick any $n \geqslant 1$, consider $\cap_{k\geqslant n} A_k$.
     - there exists some $n_0 \geqslant n$ such that $x > b_{n_0}$. i.e. $x \notin A_{n_0}$.
     - This implies $x \notin \cap_{k\geqslant n} A_k$ for any chosen $n \geqslant 1$
     - Hence, $x \notin \cup_{n\geqslant 1} \cap_{k\geqslant n} A_k$. A contradiction
- Hence, by proof of contraction, we showed $x \in (0,1]$.

Show $\liminf_{n\to\infty} A_n \supseteq (0,1]$

- Let $x \in (0,1]$
- Also consider two cases
  1) Case 1: interior case, $x \in (0,1)$.
     - Since $a_n \to 0$ and $b_n \to 1$, there exists n_0 such that for every $k \geqslant n_0$, $x \in \{y : a_k \leqslant y \leqslant b_k\} = A_k$.
     - Hence, $x \in \cap_{k\geqslant n_0} A_k$
     - Hence, $x \in \cup_{n\geqslant 1} \cap_{k\geqslant n} A_k = \liminf_{n\to\infty} A_k$.
  2) Case 2: boundary case, $x = 1$
     - Since $\lim_{n\to\infty} a_n = 0$, there exists $n_0 \in \mathbb{Z}_{\geqslant 1}$ such that $a_k < 1$ for all $k \geqslant n_0$; and $b_k > 1$ for every $k \geqslant 1$. Hence $1 \in A_k$ for all $k \geqslant n_0$,
     - Hence, $1 \in \bigcap_{k\geqslant n_0} A_k \subseteq \bigcup_{n\geqslant 1} \bigcap_{k\geqslant n} A_k = \liminf_{n\to\infty} A_n$
- Hence, we showed $x \in \liminf_{n\to\infty} A_n$.

Problem 1.9.19

For sets A, B show

$$\mathbf{1}_{A\cup B} = \mathbf{1}_A \vee \mathbf{1}_B$$
$$\mathbf{1}_{A\cap B} = \mathbf{1}_A \wedge \mathbf{1}_B$$

Solution:

Show $\mathbf{1}_{A\cup B} = \mathbf{1}_A \vee \mathbf{1}_B$

Case 1: $\omega \notin A \cup B$ First note that if $\omega \notin A \cup B$, then $\omega \notin A$ and $\omega \notin B$. Hence,

$$\mathbf{1}_{A\cup B}(\omega) = \mathbf{1}_A(\omega) = \mathbf{1}_B(\omega) = 0$$

Hence, in the case of $\omega \notin A \cup B$, we have

$$\mathbf{1}_{A\cup B} = \mathbf{1}_A \vee \mathbf{1}_B$$

Case 2: $\omega \in A \cup B$ Next we consider $\omega \in A \cup B$. We can decompose $A \cup B$ into disjoint sets

$$A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$$

Thus, we will have

$$\mathbf{1}_{A\cup B} = \mathbf{1}_{A\backslash B} + \mathbf{1}_{A\cap B} + \mathbf{1}_{B\backslash A}$$
$$\mathbf{1}_A = \mathbf{1}_{A\backslash B} + \mathbf{1}_{A\cap B}$$
$$\mathbf{1}_B = \mathbf{1}_{B\backslash A} + \mathbf{1}_{A\cap B}$$

Now we have 3 cases

1) $\omega \in A \backslash B$, then

- $\mathbf{1}_{A \cup B}(\omega) = \mathbf{1}_{A \backslash B}(\omega) + \mathbf{1}_{A \cap B}(\omega) + \mathbf{1}_{B \backslash A}(\omega) = 1 + 0 + 0 = 1$
- $\mathbf{1}_{A}(\omega) = \mathbf{1}_{A \backslash B}(\omega) + \mathbf{1}_{A \cap B}(\omega) = 1 + 0 = 1$
- $\mathbf{1}_{B}(\omega) = \mathbf{1}_{B \backslash A}(\omega) + \mathbf{1}_{A \cap B}(\omega) = 0 + 0 = 0$
- hence, we have $\mathbf{1}_{A \cup B}(\omega) = 1 = 1 \vee 0 = \mathbf{1}_{A}(\omega) \vee \mathbf{1}_{B}(\omega)$

2) $\omega \in A \cap B$, then

- $\mathbf{1}_{A \cup B}(\omega) = \mathbf{1}_{A \backslash B}(\omega) + \mathbf{1}_{A \cap B}(\omega) + \mathbf{1}_{B \backslash A}(\omega) = 0 + 1 + 0 = 1$
- $\mathbf{1}_{A}(\omega) = \mathbf{1}_{A \backslash B}(\omega) + \mathbf{1}_{A \cap B}(\omega) = 0 + 1 = 1$
- $\mathbf{1}_{B}(\omega) = \mathbf{1}_{B \backslash A}(\omega) + \mathbf{1}_{A \cap B}(\omega) = 0 + 1 = 1$
- hence, we have $\mathbf{1}_{A \cup B}(\omega) = 1 = 1 \vee 1 = \mathbf{1}_{A}(\omega) \vee \mathbf{1}_{B}(\omega)$

3) $\omega \in B \backslash A$, then

- $\mathbf{1}_{A \cup B}(\omega) = \mathbf{1}_{A \backslash B}(\omega) + \mathbf{1}_{A \cap B}(\omega) + \mathbf{1}_{B \backslash A}(\omega) = 0 + 0 + 1 = 1$
- $\mathbf{1}_{A}(\omega) = \mathbf{1}_{A \backslash B}(\omega) + \mathbf{1}_{A \cap B}(\omega) = 0 + 0 = 0$
- $\mathbf{1}_{B}(\omega) = \mathbf{1}_{B \backslash A}(\omega) + \mathbf{1}_{A \cap B}(\omega) = 1 + 0 = 1$
- hence, we have $\mathbf{1}_{A \cup B}(\omega) = 1 = 0 \vee 1 = \mathbf{1}_{A}(\omega) \vee \mathbf{1}_{B}(\omega)$

Since ω is arbitrary. Hence, we showed $\mathbf{1}_{A \cup B} = \mathbf{1}_{A} \vee \mathbf{1}_{B}$.

Show $\mathbf{1}_{A \cap B} = \mathbf{1}_{A} \wedge \mathbf{1}_{B}$

Case 1: $\omega \notin A \cap B$ Firstly, still consider if $\omega \notin A \cap B$ where we have $\mathbf{1}_{A \cap B}(\omega) = 0$. Then there are 3 case: (1) $\omega \notin A \cup B$, (2) $\omega \in A \backslash B$, (3) $\omega \in B \backslash A$. The indicator functions $\mathbf{1}_{A}, \mathbf{1}_{B}$ of these 3 cases are

1) $\omega \notin A \cup B \implies \mathbf{1}_{A}(\omega) = 0, \mathbf{1}_{B}(\omega) = 0 \implies \mathbf{1}_{A \cap B}(\omega) = 0 = \mathbf{1}_{A}(\omega) \wedge \mathbf{1}_{B}(\omega)$

2) $\omega \in A \backslash B \implies \mathbf{1}_{A}(\omega) = 1, \mathbf{1}_{B}(\omega) = 0 \implies \mathbf{1}_{A \cap B}(\omega) = 0 = 1 \wedge 0 = \mathbf{1}_{A}(\omega) \wedge \mathbf{1}_{B}(\omega)$

3) $\omega \in B \backslash A \implies \mathbf{1}_{A}(\omega) = 0, \mathbf{1}_{B}(\omega) = 1 \implies \mathbf{1}_{A \cap B}(\omega) = 0 = 0 \wedge 1 = \mathbf{1}_{A}(\omega) \wedge \mathbf{1}_{B}(\omega)$

Since $\omega \notin A \cap B$ is arbitrary. Hence, we proved the $\mathbf{1}_{A \cap B} = \mathbf{1}_{A} \wedge \mathbf{1}_{B}$ for $\omega \notin A \cap B$.

Case 2: $\omega \in A \cap B$ Now we consider the case for $\omega \in A \cap B$. If $\omega \in A \cap B$, then $\omega \in A$ and $\omega \in B$. Hence, their indicator functions are

$$\mathbf{1}_{A \cap B}(\omega) = \mathbf{1}_{A}(\omega) = \mathbf{1}_{B}(\omega) = 1$$

Thus, we have

$$\mathbf{1}_{A \cap B}(\omega) = 1 = 1 \wedge 1 = \mathbf{1}_{A}(\omega) \wedge \mathbf{1}_{B}(\omega)$$

Since $\omega \in A \cap B$ is arbitrary, hence we complete the proof.

2 Homework 2

1.9 - 10, 14, 16, 17, 18.

Problem 1.9.10

Check that $A_n \to A$ iff $\mathbf{1}_{A_n} \to \mathbf{1}_A$ pointwise.
Solution: Using

$$\liminf_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\liminf_{n\to\infty} A_n}$$
$$\limsup_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\limsup_{n\to\infty} A_n}$$

$(\implies)$ Suppose $A_n \to A$. We have the following

$$A = \lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$$
$$\mathbf{1}_A(\omega) = \mathbf{1}_{\lim_{n\to\infty} A_n}(\omega) = \mathbf{1}_{\limsup_{n\to\infty} A_n}(\omega) = \mathbf{1}_{\liminf_{n\to\infty} A_n}(\omega), \quad \forall\omega$$
$$\lim_{n\to\infty} \mathbf{1}_{A_n}(\omega) = \limsup_{n\to\infty} \mathbf{1}_{A_n}(\omega) = \liminf_{n\to\infty} \mathbf{1}_{A_n}(\omega), \quad \forall\omega$$

Pack up the equality we have, we got

$$\mathbf{1}_A(\omega) = \lim_{n\to\infty} \mathbf{1}_{A_n}(\omega), \qquad \forall\omega$$

This proves the forward direction.
$(\impliedby)$ Conversely, suppose $\mathbf{1}_{A_n} \to \mathbf{1}_A$ pointwise. Then for every ω the limit of $\mathbf{1}_{A_n}(\omega)$ exists and equals $\mathbf{1}_A(\omega)$, so

$$\mathbf{1}_{\limsup_{n\to\infty} A_n} = \limsup_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_A = \liminf_{n\to\infty} \mathbf{1}_{A_n} = \mathbf{1}_{\liminf_{n\to\infty} A_n} \qquad \text{pointwise}$$

using the two identities quoted at the start. Since a set is determined by its indicator function, $\limsup_{n\to\infty} A_n = A = \liminf_{n\to\infty} A_n$, that is, $A_n \to A$.

Problem 1.9.14

Suppose that $\mathcal{A}_n$ are fields satisfying $\mathcal{A}_n \subset \mathcal{A}_{n+1}$. Show that $\cup_n \mathcal{A}_n$ is a field. (But see also the next problem.)
Solution: Checking the 3 axioms of a field.

(i) $\Omega \in \cup_n \mathcal{A}_n$ Since $\Omega \in \mathcal{A}_n$ for all n (because all $\mathcal{A}_n$ are fields), thus $\Omega \in \cup_n \mathcal{A}_n$

(ii) Closed under complement Let $A \in \cup_n \mathcal{A}_n$. Because $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, hence there exists a $n_0 \in \mathbb{Z}_{\geqslant 1}$ such that for all $m \geqslant n_0$, $A \in \mathcal{A}_m$. Because these $\mathcal{A}_m$ are fields, hence $\Omega\backslash A \in \mathcal{A}_m$ for all $m \geqslant n_0$. Hence, $\Omega\backslash A \in \cup_n \mathcal{A}_n$.

(iii) Closed under finite union Let $A_1, \ldots, A_k \in \cup_n \mathcal{A}_n$. Still because $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, for every $i = 1, \ldots, k$, there exists a $n_i \in \mathbb{Z}_{\geqslant 1}$ such that for all $m \geqslant n_i$, $A_i \in \mathcal{A}_m$. Denote $n_0 = \max_i n_i$. Hence, $A_1, \ldots, A_k \in \mathcal{A}_m$ for all $m \geqslant n_0$. Because $\mathcal{A}_m$'s are all fields, we have $\cup_{i=1}^k A_i \in \mathcal{A}_m$ for all $m \geqslant n_0$. Hence, $\cup_{i=1}^k A_i \in \cup_n \mathcal{A}_n$.

Problem 1.9.16

Suppose $\mathcal{A}$ is a class of subsets of Ω such that

- $\Omega \in \mathcal{A}$
- $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$
- $\mathcal{A}$ is closed under finite disjoint unions.

Show A does not have to be a field. Hint: Try $\Omega = \{1, 2, 3, 4\}$ and let $\mathcal{A}$ be the field generated by two point subsets of Ω.
Solution: Let $\mathcal{A} = \{\Omega, \varnothing, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\}$. It is clear that $\Omega \in \mathcal{A}$, and $\mathcal{A}$ is closed under complement and finite union of disjoint sets.

However, $\mathcal{A}$ is not closed under finite union. Say $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\} \notin \mathcal{A}$. Hence, $\mathcal{A}$ is not a field.

Problem 1.9.17

Prove $\liminf_{n\to\infty} A_n = \{\omega : \lim_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 1\}$.
Solution: Recall that

$$\begin{aligned}\liminf_{n\to\infty} A_n &= \{\omega : \omega \in A_n \text{ for all } n \text{ except a finite number}\} \\ &= \left\{\omega : \sum_n \mathbf{1}_{A_n^c}(\omega) < \infty\right\}\end{aligned}$$

Hence, there exists some $n_0 \in \mathbb{Z}_{\geqslant 1}$ such that for all $m \geqslant n_0$, $\mathbf{1}_{A_m^c} = 0$. Equivalently, $\mathbf{1}_{A_m} = 1$. Hence, for any $\omega \in \liminf_{n\to\infty} A_n$, we have $\lim_{n\to\infty} \mathbf{1}_{A_n}(\omega) = 1$.

$$\left\{\omega : \sum_n \mathbf{1}_{A_n^c}(\omega) < \infty\right\} = \left\{\omega : \lim_{n\to\infty} \mathbf{1}_{A_n^c} = 0\right\} = \left\{\omega : \lim_{n\to\infty} \mathbf{1}_{A_n} = 1\right\}$$

Problem 1.9.18

Suppose $\mathcal{A}$ is a class of sets containing Ω and satisfying

$$A, B \in \mathcal{A} \implies A \backslash B = AB^c \in \mathcal{A}$$

Show $\mathcal{A}$ is a field.
Solution: Check the 3 field axioms.

(i) $\Omega \in \mathcal{A}$. This is given by the sentence "Suppose $\mathcal{A}$ is a class of sets containing Ω".

(ii) Closed under complement Let $B \in \mathcal{A}$. We know $\Omega \in \mathcal{A}$. Then let $A = \Omega$ and $B = B$. Thus $\Omega \backslash B \in \mathcal{A}$.

(iii) Closed under finite union Let $A, B \in \mathcal{A}$. WTS $A \cup B \in \mathcal{A}$. By (ii), we have $A^c, B^c \in \mathcal{A}$. Thus, by assumption, we have $A^c \backslash B = A^c \cap B^c \in \mathcal{A}$. Hence, again by (ii), we have $A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$. Hence, we completes the proof.

3 Homework 3

HW3 (due on 09/18): 1.9 - 21, 22, 31(hard one!), 41.

Problem 1.9.21

Suppose $\mathcal{A}$ is a field and suppose also that $\mathcal{A}$ has the property that it is closed under countable disjoint unions. Show $\mathcal{A}$ is a σ-field.

Proof. Let $A_1, \ldots, A_n, \ldots \in \mathcal{A}$. Define new sets B_i as

$$B_1 = A_1, \quad B_2 = A_1 \cup A_2, \quad B_3 = A_1 \cup A_2 \cup A_3, \ldots$$

Under this construction, we have B_n is an increasing sequence of sets. And for every $n \in \mathbb{Z}_{\geqslant 1}$

$$\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k$$

Next, we can use disjointification, we have

$$C_1 = B_1, \quad C_2 = B_2 \backslash B_1, \quad C_3 = B_3 \backslash B_2, \ldots$$

where $C_i \cap C_j = \varnothing$ by construction. Note that each $B_n \in \mathcal{A}$ because a field is closed under finite unions, and hence each $C_k = B_k \cap B_{k-1}^c \in \mathcal{A}$ as well, by closure under complement and finite intersection. Moreover, for every $n \in \mathbb{Z}_{\geqslant 1}$

$$\bigcup_{k=1}^{n} C_k = \bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} A_k$$

Under the assumption that $\mathcal{A}$ is closed under countable disjoint union, we have

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} C_k \in \mathcal{A}$$

Hence, we proved $\mathcal{A}$ is a σ-algebra. □

Problem 1.9.22

Let Ω be a non-empty set and let $\mathcal{C}$ be all one point subsets. Show that

$$\sigma(\mathcal{C}) = \{A \subset \Omega : A \text{ is countable}\} \bigcup \{A \subset \Omega : A^c \text{ is countable}\}$$

Solution: Here $\mathcal{C}$ is defined as

$$\mathcal{C} := \{\{x\} : x \in \Omega\}$$

Note that

$$\begin{aligned} \mathscr{B} &:= \{A \subset \Omega : \text{ either A is a countable set, or } A^c \text{ is a countable set}\} \\ &\equiv \{A \subset \Omega : A \text{ is countable}\} \bigcup \{A \subset \Omega : A^c \text{ is countable}\} \end{aligned}$$

which we have already proved in lecture that it is a σ-field.

Show $\mathcal{C} \subseteq \mathscr{B}$. Clearly any $\{x\}$ is countable, hence $\{x\} \in \mathscr{B}$ for any $\{x\} \in \mathcal{C}$. Thus, $\mathcal{C} \subset \mathscr{B}$. Then by the definition of $\sigma(\mathcal{C})$ being the smallest σ-field containing $\mathcal{C}$, we have $\sigma(\mathcal{C}) \subseteq \mathscr{B}$.

Let $A \in \mathscr{B}$. Now two cases. If A is countable, then we can express A as a countable union of singletons.

$$A = \bigcup_{x \in A} \{x\} \in \sigma(\mathcal{C})$$

If A is uncountable, then by definition of $\mathscr{B}$, we have A^c is countable. By the same reasoning, we can express A^c as a countable union of singletons.

$$A^c = \bigcup_{x \in A^c} \{x\} \in \sigma(\mathcal{C})$$

Since $\sigma(\mathcal{C})$ is a σ-field, it is closed under complement, so $A = (A^c)^c \in \sigma(\mathcal{C})$.

Hence, we conclude $\mathscr{B} \subseteq \sigma(\mathcal{C})$.

Thus, we conclude $\mathscr{B} = \sigma(\mathcal{C})$.

Problem 1.9.31

Suppose Ω is uncountable and let $\mathcal{G}$ be the σ-field consisting of sets A such that either A is countable or A^c is countable. Show $\mathcal{G}$ is NOT countably generated.

(Hint: If $\mathcal{G}$ were countably generated, it would be generated by a countable collection of one point sets.)

In fact, if $\mathcal{G}$ is the σ-field of subsets of Ω consisting of the countable and co-countable sets, $\mathcal{G}$ is countably generated iff Ω is countable.

Solution: Here

$$\mathcal{G} = \{A \subseteq \Omega : \text{ either A is a countable set, or } A^c \text{ is a countable set}\}$$

Assume by contradiction that $\mathcal{G}$ is countably generated, and let $\mathcal{A} := \{A_n : n \in \mathbb{Z}_{\geqslant 1}\}$ be this countable family of sets, i.e.

$$\mathcal{G} \equiv \sigma(\{A_n : n \in \mathbb{Z}_{\geqslant 1}\}) = \sigma(\mathcal{A})$$

By definition of $\mathcal{G}$, we have for each $n \geqslant 1$, either A_n is countable or A_n^c is countable.

Following the hint, our plan is: let $C \subsetneq \Omega$ be a countable subset of Ω and show $\mathcal{G} = \sigma(\mathcal{C})$ where $\mathcal{C} = \{\{x\} : x \in C\}$; a singleton outside C will then produce a contradiction. Let's consider a different generating set of $\mathcal{G}$. Define sets another family of sets $\mathcal{B} = \{B_n : n \in \mathbb{Z}_{\geqslant 1}\}$ by

$$B_n = \begin{cases} A_n & \text{if } A_n \text{ is countable} \\ A_n^c & \text{if } A_n \text{ is uncountable} \end{cases}$$

and see that

$$\mathcal{G} \equiv \sigma(\{A_n : n \in \mathbb{Z}_{\geqslant 1}\}) = \sigma(\{B_n : n \in \mathbb{Z}_{\geqslant 1}\})$$

simply by definition of generated σ-field. We choose $C = \cup_n B_n$, which is a countable union of countable sets, so of course still countable.

WTS: $\sigma(\mathcal{B}) = \sigma(\mathcal{C})$. Take a generating set $B_n \in \sigma(\mathcal{B})$. Because it is countable and $B_n \subset C$, so we can write it as a countable union of singletons $\{x\} \subset \mathcal{C}$.

$$B_n = \bigcup_{x \in B_n} \{x\}$$

Hence, $B_n \in \sigma(\mathcal{C})$. Conversely, consider $\{x\} \in \mathcal{C}$. Clearly $\{x\}$ is countable. Hence, $\{x\} \in \mathcal{G} = \sigma(\mathcal{B})$. Hence, $\mathcal{C} \subseteq \mathcal{G}$. Hence, $\sigma(\mathcal{C}) \subseteq \mathcal{G} = \sigma(\mathcal{B})$.

Hence, we have $\mathcal{G} = \sigma(\mathcal{B}) = \sigma(\mathcal{C})$.

Consider $p \in \Omega \backslash C$. Clearly $\{p\}$ is countable, hence $\{p\} \in \mathcal{G} = \sigma(\mathcal{C})$. But the claim is $\{p\} \notin \sigma(\mathcal{C})$. To see this, we compute $\sigma(\mathcal{C})$ explicitly:

$$\sigma(\mathcal{C}) = \{A \subseteq \Omega : A \subseteq C \quad \text{or} \quad \Omega \backslash A \subseteq C\}$$

Indeed, the collection on the right contains every generator $\{x\}$, $x \in C$. It is a σ-field: it is closed under complement by the symmetry of the two conditions, and for a countable union, if every $A_i \subseteq C$ then $\bigcup_i A_i \subseteq C$, while if some A_{i_0} has $\Omega \backslash A_{i_0} \subseteq C$ then $\Omega \backslash \bigcup_i A_i \subseteq \Omega \backslash A_{i_0} \subseteq C$. And it is the smallest such: any σ-field containing $\mathcal{C}$ contains every $A \subseteq C$ (a countable union of generators, since C is countable), hence also every A with $\Omega \backslash A \subseteq C$, as the complement of such a set.

Now take $p \in \Omega \backslash C$. Then $\{p\} \nsubseteq C$ because $p \notin C$; and $\Omega \backslash \{p\} \nsubseteq C$, since otherwise $\Omega \subseteq C \cup \{p\}$ would be countable while Ω is uncountable. So $\{p\}$ satisfies neither condition, i.e. $\{p\} \notin \sigma(\mathcal{C})$. Hence, a contradiction.

Problem 1.9.41

A monotone class $\mathcal{M}$ is a non-empty collection of subsets of Ω closed under monotone limits; that is, if $A_n \uparrow$ and $A_n \in \mathcal{M}$, then $\lim_{n \to \infty} A_n = \cup_n A_n \in M$ and if $A_n \downarrow$ and $A_n \in \mathcal{M}$, then $\lim_{n \to \infty} A_n = \cap A_n \in M$.

Show that a σ-field is a field that is also a monotone class and conversely, a field that is a monotone class is a σ-field.

Solution:

Show σ-field is a field + monotone class Let $\mathcal{A}$ be a σ-field. Then clearly $\mathcal{A}$ is a field. It is also clear that $\mathcal{A}$ is a monotone class. Let $A_1 \subseteq A_2 \subseteq \ldots$ be an increasing subsets in $\mathcal{A}$. Then

$$\bigcup_{n=1}^{N} A_n = A_N$$

Hence,

$$\lim_{N \to \infty} A_N = \lim_{N \to \infty} \bigcup_{n=1}^{N} A_n = \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$$

Similarly for decreasing sequence of subsets $A_1 \supseteq A_2 \supseteq \ldots$. Because $A_i \in \mathcal{A}$, then $A_i^c \in \mathcal{A}$ by axiom of σ-field. Hence $\cup_{n \geq 1} A_n^c \in \mathcal{A}$ again by axiom of σ-field. Lastly, use the fact that

$$\bigcap_{n=1}^{N} A_n = A_N$$

closure under complement again, we have

$$\lim_{N\to\infty} A_N = \lim_{N\to\infty} \bigcap_{n=1}^{N} A_n = \bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{A}$$

Hence we proved σ-field is a monotone class.

Show field + monotone class is a σ-field So let $\mathcal{A}$ be a field and monotone class. Hence $\mathcal{A}$ satisfies the first two axioms. We only need to show closure under countable union.

Let $A_1, A_2, \ldots \in \mathcal{A}$. Define

$$B_n = \bigcup_{k=1}^{n} A_k \in \mathcal{A}$$

Hence, we have

$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k$$

which makes B_n an increasing sequence of sets. Then by definition of monotone class, we have

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k = \lim_{n\to\infty} B_n \in \mathcal{A}$$

Hence, it is a σ-algebra.

There is another proof using decreasing sequence definition of monotone class. See that showing $\cup_n A_n \in \mathcal{A}$ is equivalent to show $(\cup_n A_n)^c = \cap_n A_n^c \in \mathcal{A}$ by using closure under complement of a field. Then define

$$B_n = \bigcap_{k=1}^{n} A_k^c \in \mathcal{A}$$

which is a decreasing sequence of sets. We have

$$\bigcap_{k=1}^{\infty} A_k^c = \bigcap_{k=1}^{\infty} B_k = \lim_{n\to\infty} B_n \in \mathcal{A}$$

which completes the proof.

4 Homework 4

HW4 (due on 09/25): 1.9 - 33(i), 34, 44; 2.6 - 1(b).

Problem 1.9.33 (The extended real line)

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be the extended or closed real line with the points $-\infty$ and ∞ added. The Borel sets $\mathscr{B}(\overline{\mathbb{R}})$ is the σ-field generated by the sets $[-\infty, x], x \in \mathbb{R}$, where $[-\infty, x] = \{-\infty\} \cup (-\infty, x]$. Show $\mathscr{B}(\overline{\mathbb{R}})$ is also generated by the following collections of sets

1) $[-\infty, x), x \in \mathbb{R}$
2) $(x, \infty], x \in \mathbb{R}$
3) all finite intervals and $\{-\infty\}$ and $\{\infty\}$.

Now think of $\overline{\mathbb{R}} = [-\infty, \infty]$ as homeomorphic in the topological sense to $[-1, 1]$ under the transformation

$$x \longmapsto \frac{x}{1 - |x|}$$

from $[-1, 1]$ to $[-\infty, \infty]$. (This transformation is designed to stretch the finite interval onto the infinite interval.) Consider the usual topology on $[-1, 1]$ and map it onto a topology on $[-\infty, \infty]$. This defines a collection of open sets on $[-\infty, \infty]$ and these open sets can be used to generate a Borel σ-field. How does this σ-field compare with $\mathscr{B}(\overline{\mathbb{R}})$ described above?

Part 1

Denote

$$\begin{aligned}
\mathcal{F}_0 &:= \{[-\infty, x] : x \in \mathbb{R}\} \\
\mathcal{F}_1 &:= \{[-\infty, x) : x \in \mathbb{R}\} \\
\mathcal{F}_2 &:= \{(x, \infty] : x \in \mathbb{R}\} \\
\mathcal{F}_3 &:= \{[a, b] : a, b \in \mathbb{R}\} \cup \{-\infty\} \cup \{\infty\}
\end{aligned}$$

and for $\mathcal{F}_3$, we choose the closed bracket, but this can be closed, half-open & half-closed. The question defines $\mathscr{B}(\overline{\mathbb{R}})$ as

$$\mathscr{B}(\overline{\mathbb{R}}) := \sigma(\mathcal{F}_0)$$

Show $\mathscr{B}(\overline{\mathbb{R}})$ is generated by the sets in $\mathcal{F}_1$ We want to show $\sigma(\mathcal{F}_1) = \sigma(\mathcal{F}_0)$. See that

$$[-\infty, x] = \bigcap_{n \geqslant 1} \underbrace{[-\infty, x + \frac{1}{n})}_{\in \mathcal{F}_1} \in \sigma(\mathcal{F}_1)$$

and

$$[-\infty, x) = \bigcup_{n \geqslant 1} \underbrace{[-\infty, x - \frac{1}{n}]}_{\in \mathcal{F}_0} \in \sigma(\mathcal{F}_0)$$

Show $\mathscr{B}(\overline{\mathbb{R}})$ is generated by the sets in $\mathcal{F}_2$ We want to show $\sigma(\mathcal{F}_2) = \sigma(\mathcal{F}_0)$. See that

$$(x, \infty] = \underbrace{[-\infty, x]}_{\in \mathcal{F}_0}{}^c \in \sigma(\mathcal{F}_0)$$

and

$$[-\infty, x] = \underbrace{(x, \infty]}_{\in \mathcal{F}_2}{}^c \in \sigma(\mathcal{F}_2)$$

Show $\mathscr{B}(\overline{\mathbb{R}})$ is generated by the sets in $\mathcal{F}_3$ We want to show $\sigma(\mathcal{F}_3) = \sigma(\mathcal{F}_0)$. See that

$$[-\infty, x] = \bigcup_{n \geqslant 1} \underbrace{[-n, x]}_{\in \mathcal{F}_3} \cup \underbrace{\{-\infty\}}_{\in \mathcal{F}_3} \in \sigma(\mathcal{F}_3)$$

and

$$\begin{aligned}
[a, b] &= [-\infty, b] \cap \bigcap_{n \geqslant 1} \big(a - \frac{1}{n}, \infty\big] = \underbrace{[-\infty, b]}_{\in \mathcal{F}_0} \cap \bigcap_{n \geqslant 1} \underbrace{\big[-\infty, a - \frac{1}{n}\big]^c}_{\in \mathcal{F}_0} \in \sigma(\mathcal{F}_0) \\
\{\infty\} &= \bigcap_{n \geqslant 1} \big(n, \infty\big] = \bigcap_{n \geqslant 1} \underbrace{\big[-\infty, n\big]}_{\in \mathcal{F}_0}{}^c \in \sigma(\mathcal{F}_0) \\
\{-\infty\} &= \bigcap_{n \geqslant 1} \underbrace{\big[-\infty, -n\big]}_{\in \mathcal{F}_0} \in \sigma(\mathcal{F}_0)
\end{aligned}$$

Part 2

The Borel σ-fields are clearly not equal. But since there is a homeomorphism between two topological space, there will be a one-to-one correspondence between Borel sets from different topological spaces. Hence there is an isomorphism between these two Borel σ-fields induced by that homeomorphism.

Problem 1.9.34

Suppose $\mathscr{B}$ is a σ-field of subsets of Ω and suppose $A \notin \mathscr{B}$. Show that $\sigma(\mathscr{B} \cup \{A\})$, the smallest σ-field containing both $\mathscr{B}$ and A consists of sets of the form

$$AB \cup A^c B', \qquad B, B' \in \mathscr{B}$$

Solution: Define

$$\mathcal{F} := \{AB \cup A^c B' : B, B' \in \mathscr{B}\}$$

Want to show $\mathcal{F} = \sigma(\mathscr{B} \cup \{A\})$.

Show $\mathcal{F} \subseteq \sigma(\mathscr{B} \cup \{A\})$. First of all, it is clear that for any $B, B' \in \mathscr{B}$, the set $AB \cup A^c B' \in \sigma(\mathscr{B} \cup \{A\})$ because $AB \cup A^c B'$ is a finite union and intersection of elements in $\sigma(\mathscr{B} \cup \{A\})$. Hence, $\mathcal{F} \subseteq \sigma(\mathscr{B} \cup \{A\})$.

Show $\mathscr{B} \cup \{A\} \subseteq \mathcal{F}$. Given $AB \cup A^cB'$, we can easily recover all atoms $C \in \mathscr{B}$ and the set A. For arbitrary $C \in \mathscr{B}$, choose $B = B' = C \in \mathscr{B}$. Then

$$AB \cup A^cB' = AC \cup A^cC = C \cap (A \cup A^c) = C \cap \Omega = C$$

And for the set A, choose $B = \Omega$ and $B' = \varnothing$. Then

$$AB \cup A^cB' = A\Omega \cup A^c\varnothing = A$$

Hence, we conclude $\mathscr{B} \cup \{A\} \subset \mathcal{F}$.

Show $\mathcal{F}$ is a σ-field By choosing $B = B' = \Omega$, we have $A\Omega \cup A^c\Omega = A \cup A^c = \Omega$. Hence, $\Omega \in \mathcal{F}$.

If $AB \cup A^cB' \in \mathcal{F}$, then

$$\begin{aligned}
(AB \cup A^cB')^c &= (AB)^c \cap (A^cB')^c = (A^c \cup B^c) \cap (A \cup (B')^c) \\
&= \underbrace{(A \cap (A^c \cup B^c))}_{=AB^c} \cup \underbrace{((B')^c \cap (A^c \cup B^c))}_{=A^c(B')^c \cup B^c(B')^c} \\
&= AB^c \cup A^c(B')^c \cup B^c(B')^c \\
&= AB^c \cup A^c(B')^c \cup B^c(B')^c\Omega \\
&= AB^c \cup A^c(B')^c \cup B^c(B')^c(A \cup A^c) \\
&= AB^c \cup A^c(B')^c \cup (AB^c(B')^c \cup A^cB^c(B')^c) \\
&= A\underbrace{(B^c \cup B^c(B')^c)}_{\in\mathscr{B}} \cup A^c\underbrace{((B')^c \cup B^c(B')^c)}_{\in\mathscr{B}} \\
&\in \mathcal{F}
\end{aligned}$$

If $AB_n \cup A^cB_n' \in \mathcal{F}$ for $n \geqslant 1$. Consider

$$\begin{aligned}
\bigcup_{n\geqslant 1}(AB_n \cup A^cB_n') &= \left(\bigcup_{n\geqslant 1} AB_n\right) \cup \left(\bigcup_{n\geqslant 1} A^cB_n'\right) \\
&= \left(A\underbrace{\bigcup_{n\geqslant 1} B_n}_{\in\mathscr{B}}\right) \cup \left(A^c\underbrace{\bigcup_{n\geqslant 1} B_n'}_{\in\mathscr{B}}\right) \\
&\in \mathcal{F}
\end{aligned}$$

Hence, we proved $\mathcal{F}$ is a σ-algebra.

Conclude $\mathcal{F} = \sigma(\mathscr{B} \cup \{A\})$ Because $\mathcal{F}$ is a σ-algebra, and $\mathcal{F} \supset \mathscr{B} \cup \{A\}$. Hence by the definition of the smallest σ-algebra, we have $\mathcal{F} \supseteq \sigma(\mathscr{B} \cup \{A\})$. But we also showed $\mathcal{F} \subseteq \sigma(\mathscr{B} \cup \{A\})$. Hence, we conclude $\mathcal{F} = \sigma(\mathscr{B} \cup \{A\})$.

Problem 1.9.44

Let $\mathcal{A}$ be a field of subsets of Ω and define

$$\bar{\mathcal{A}} = \{A \subset \Omega : \exists\ A_n \in \mathcal{A} \text{ and } A_n \to A\}.$$

Show $\bar{\mathcal{A}} \subset \mathcal{A}$ and $\bar{\mathcal{A}}$ is a field.

Show $\bar{\mathcal{A}} \subset \mathcal{A}$

It is clear that $\mathcal{A} \subset \bar{\mathcal{A}}$. Let $A \in \mathcal{A}$. Then choose $A_n = A \in \mathcal{A}$, we have $A_n \to A$. Thus, $A \in \bar{\mathcal{A}}$.

And if we can show $\bar{\mathcal{A}} \subset \mathcal{A}$, then we will have $\mathcal{A} = \bar{\mathcal{A}}$. Thus $\bar{\mathcal{A}}$ is a field because $\mathcal{A}$ is a field.

The stated inclusion $\bar{\mathcal{A}} \subset \mathcal{A}$ is false. That shortcut is not available, because for a general field $\mathcal{A}$ the inclusion $\bar{\mathcal{A}} \subset \mathcal{A}$ fails. Only $\mathcal{A} \subset \bar{\mathcal{A}}$ survives, and the second half of the exercise has to be proved on its own, as is done in the next subsection.

Here is a counterexample. Take $\Omega = \mathbb{R}$ and let

$$\mathcal{A} := \{A \subset \mathbb{R} : A \text{ is finite, or } A^c \text{ is finite}\}$$

be the co-finite class, i.e. the class $\mathcal{F}_0$ of Problem 2.6.1 below. It is a field: $\varnothing$ is finite so $\Omega \in \mathcal{A}$; the defining condition is symmetric in A and A^c, so $\mathcal{A}$ is closed under complementation; and if $A, B \in \mathcal{A}$ then either both are finite, in which case $A \cup B$ is finite, or one of them, say A, has A^c finite, in which case $(A \cup B)^c = A^c \cap B^c \subseteq A^c$ is finite. Now put

$$A_n := \{1, 2, \ldots, n\} \in \mathcal{A}, \qquad n \geqslant 1$$

This sequence is monotone increasing, hence it converges, with

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcup_{n\geqslant 1} A_n = \mathbb{Z}_{\geqslant 1}$$

So $\mathbb{Z}_{\geqslant 1} \in \bar{\mathcal{A}}$. But $\mathbb{Z}_{\geqslant 1}$ is infinite and its complement $\mathbb{R}\backslash\mathbb{Z}_{\geqslant 1}$ is infinite too, so $\mathbb{Z}_{\geqslant 1} \notin \mathcal{A}$. Hence $\bar{\mathcal{A}} \not\subset \mathcal{A}$.

The same thing happens for the field of finite disjoint unions of intervals $(a, b] \subset (0, 1]$: there $(0, \frac{1}{2} - \frac{1}{n}] \uparrow (0, \frac{1}{2})$, and $(0, \frac{1}{2})$ is not a finite union of half-open intervals.

What goes wrong is exactly the passage from finite to countable operations. If $\mathcal{A}$ were a σ-field, then for any $A_n \in \mathcal{A}$ with $A_n \to A$ we would get

$$A = \liminf_{n\to\infty} A_n = \bigcup_{n\geqslant 1} \bigcap_{k\geqslant n} A_k \in \mathcal{A}$$

and the inclusion would hold. For a mere field the sets $\bigcap_{k\geqslant n} A_k$ need not belong to $\mathcal{A}$, and the counterexample above shows the conclusion genuinely fails.

So the two assertions that are true, and that are proved here, are $\mathcal{A} \subset \bar{\mathcal{A}}$ above and "$\bar{\mathcal{A}}$ is a field" next.

Show $\bar{\mathcal{A}}$ is a field.

Auxiliary results 1 $(\liminf_{n\to\infty} A_n) \cup (\liminf_{n\to\infty} B_n) \subseteq (\liminf_{n\to\infty} A_n \cup B_n)$

Suppose $\omega \in (\liminf_{n\to\infty} A_n) \cup (\liminf_{n\to\infty} B_n)$. It means when n gets larger and larger, either $\omega \in A_n$ eventually keeps happening, or $\omega \in B_n$ eventually keeps happening. This means $\omega \in A_n \cup B_n$ will eventually keeps happening. Hence, $\omega \in \liminf_{n\to\infty} A_n \cup B_n$.

Auxiliary results 2 $(\limsup_{n\to\infty} A_n) \cup (\limsup_{n\to\infty} B_n) \supseteq (\limsup_{n\to\infty} A_n \cup B_n)$

Suppose $\omega \in \limsup_{n\to\infty} A_n \cup B_n$. Then $\omega \in A_n \cup B_n$ will happens infinitely often. Which means either $\omega \in A_n$ will happen infinitely often, or $\omega \in B_n$ will happens infinitely often. Hence, $\omega \in (\limsup_{n\to\infty} A_n) \cup (\limsup_{n\to\infty} B_n)$.

Auxiliary results 3 If $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, then $\lim_{n\to\infty} A_n \cup B_n = A \cup B$. Use previous two auxiliary results, we have

$$\begin{aligned} A \cup B &= (\liminf_{n\to\infty} A_n) \cup (\liminf_{n\to\infty} B_n) \\ &\subseteq \liminf_{n\to\infty} A_n \cup B_n \\ &\subseteq \limsup_{n\to\infty} A_n \cup B_n \\ &\subseteq (\limsup_{n\to\infty} A_n) \cup (\limsup_{n\to\infty} B_n) \\ &= A \cup B \end{aligned}$$

Hence, by "sandwich" theorem, we have

$$A \cup B = \liminf_{n\to\infty} A_n \cup B_n = \limsup_{n\to\infty} A_n \cup B_n = \lim_{n\to\infty} A_n \cup B_n$$

Proof

- $\Omega \in \bar{\mathcal{A}}$. Because $\Omega \in \mathcal{A}$. If we let $A_n = \Omega \in \mathcal{A}$, then clearly $A_n \to A$.
- Let $A \in \bar{\mathcal{A}}$. Then there exists a sequence of sets $A_n \in \mathcal{A}$ such that $\lim_{n\to\infty} A_n = A$. And $A_n^c \in \mathcal{A}$. Then

$$\begin{aligned} A^c &= \left(\lim_{n\to\infty} A_n\right)^c \\ &= \begin{cases} \left(\limsup_{n\to\infty} A_n\right)^c = \left(\bigcap_{n\geqslant 1}\bigcup_{k\geqslant n} A_k\right)^c = \bigcup_{n\geqslant 1}\bigcap_{k\geqslant n} A_k^c = \liminf_{n\to\infty} A_n^c \\ \left(\liminf_{n\to\infty} A_n\right)^c = \left(\bigcup_{n\geqslant 1}\bigcap_{k\geqslant n} A_k\right)^c = \bigcap_{n\geqslant 1}\bigcup_{k\geqslant n} A_k^c = \limsup_{n\to\infty} A_n^c \end{cases} \\ &\iff A^c = \limsup_{n\to\infty} A_n^c = \liminf_{n\to\infty} A_n^c = \lim_{n\to\infty} A_n^c \\ &\implies A^c \in \bar{\mathcal{A}} \end{aligned}$$

- Let $A, B \in \bar{\mathcal{A}}$. Then there exists sequences of sets $A_n, B_n \in \mathcal{A}$ such that $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$.
- Since $\mathcal{A}$ is a field, we have $A_n \cup B_n \in \mathcal{A}$.
- Lastly, since $A \cup B = \lim_{n\to\infty} A_n \cup B_n$. Hence, we have $A \cup B \in \bar{\mathcal{A}}$.

Problem 2.6.1

Let Ω be a non-empty set. Let $\mathcal{F}_0$ be the collection of all subsets such that either A or A^c is finite.

(a)

Show that $\mathcal{F}_0$ is a field.

- $\varnothing$ is finite, hence $\Omega \in \mathcal{F}_0$
- If $A \in \mathcal{F}_0$. Then two cases. (1) A is finite, (2) A is infinite. Case (1) means A^c is infinite, and $(A^c)^c = A$ is finite. Hence $A^c \in \mathcal{F}_0$. Case (2) means A^c finite, hence $A^c \in \mathcal{F}_0$.
- If $A_1, \ldots, A_k \in \mathcal{F}_0$. Then two cases. (1) All A_i are finite: then $\cup_{i=1}^k A_i$ is a finite union of finite sets, hence finite. (2) At least one of them, say A_1, is infinite. Then A_1^c is finite. Consider $(\cup_{i=1}^k A_i)^c = \cap_{i=1}^k A_i^c \subset A_1^c$; a subset of a finite set is finite, hence $(\cup_{i=1}^k A_i)^c$ is finite. Hence $\cup_{i=1}^k A_i \in \mathcal{F}_0$.

Information

Define for $E \in \mathcal{F}_0$ the set function P by

$$P(E) := \begin{cases} 0, & \text{if } E \text{ is finite} \\ 1, & \text{if } E^c \text{ is finite} \end{cases}$$

(b)

If Ω is countably infinite, show P is finitely additive but not σ-additive.

Show P is finitely additive Let A, B be disjoint sets. Then we have 3 different cases

1. A, B are both finite.
2. A is finite, and B is co-finite
3. A, B both co-finite

Claim 3: Note that we cannot have both arbitrary disjoint $A, B \subseteq \Omega$ being co-finite sets. Because suppose by contradiction if they are, i.e. $A^c, B^c \subseteq \Omega$ are finite. Hence, $A^c \cup B^c = (A \cap B)^c$ is finite. But since A, B are disjoint, i.e. $A \cap B = \varnothing$. Hence, $(A \cap B)^c = \Omega$ which is infinite. Hence, a contradiction.

Case 1: If A, B are both finite, then $A \cup B$ is finite. Hence

$$\begin{aligned} P(A \cup B) &= 0 \\ P(A) &= 0 \\ P(B) &= 0 \end{aligned}$$

Hence,

$$P(A \cup B) = 0 = 0 + 0 = P(A) + P(B)$$

Case 2: If A is finite, and B is co-finite, then $A \cup B$ is co-finite, since $(A \cup B)^c = A^c \cap B^c \subseteq B^c$ is finite. Hence

$$\begin{aligned} P(A \cup B) &= 1 \\ P(A) &= 0 \\ P(B) &= 1 \end{aligned}$$

Hence,

$$P(A \cup B) = 1 = 1 + 0 = P(A) + P(B)$$

Show P is not σ-additive Since Ω is countable, denote elements in Ω be

$$\Omega := \{\omega_i : i \geqslant 1\}$$

Choose $A_n = \{\omega_n\}$, $n \geqslant 1$ be a sequence of disjoint subsets of Ω, which are clearly in $\mathcal{F}_0$. Because A_n is finite, hence

$$P(A_n) = 0$$

But see that

$$\bigcup_{n \geqslant 1} A_n = \Omega$$

Thus

$$P\left(\bigcup_{n \geqslant 1} A_n\right) = P(\Omega) = 1$$

Thus, not σ-additive

(c)

If Ω is uncountably infinite, show P is σ-additive on $\mathcal{F}_0$.

Solution Let $A_n \in \mathcal{F}_0$, $n \geqslant 1$ be a countable sequence of disjoint elements in $\mathcal{F}_0$. Then we have three cases

1. All A_n's are finite.
2. One of A_n is co-finite.
3. More than 2 A_n's are co-finite

Case 1: See that there can be at most finitely many of distinct non-empty A_n's that are finite. Because if there are countably many of distinct non-empty finite A_n's, then $\cup_n A_n$ is a countable union of finite set, which is countable. But $(\cup_n A_n)^c$ will not be finite since Ω is uncountable.

The only exception is that if the countably many of non-empty disjoint subsets stops at most $A_1, \ldots, A_N$ different non-empty disjoint subsets for some $N \in \mathbb{Z}_{\geqslant 1}$, i.e. we have $\cup_{n=1}^N A_n$, which is finite. And $\varnothing = A_{N+1} = \ldots$.

Then since everything is finite, we have

$$P\left(\bigcup_{n=1}^N A_n\right) = 0 = \sum_{n=1}^N 0 = \sum_{n=1}^N P\Big(A_n\Big)$$

Case 2: If one of A_n is co-finite, say A_1 is co-finite, i.e. A_1^c is finite. Then consider

$$\left(\bigcup_{n\geqslant 1} A_n\right)^c = \bigcap_{n\geqslant 1} A_n^c \subset A_1^c$$

will be finite. Hence, we have $\cup_{n\geqslant 1} A_n \in \mathcal{F}_0$, and

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 = 1 + \sum_{n=2}^{\infty} 0 = P(A_1) + \sum_{n=2}^{\infty} P(A_n)$$

Case 3: Because suppose by contradiction if there are more than 2 disjoint A_n's are co-finite. Let's just consider $n = 2$, where A_1^c, A_2^c are finite. Then the proof will be the same as Ω finite case. The induction will fail at $n = 2$.

In detail: suppose $A_1, A_2 \in \mathcal{F}_0$ are disjoint and both co-finite, i.e. both A_1^c and A_2^c are finite. Since $A_1 \cap A_2 = \varnothing$, De Morgan gives

$$\Omega = \varnothing^c = (A_1 \cap A_2)^c = A_1^c \cup A_2^c$$

which is a union of two finite sets, hence finite. This contradicts the standing assumption that Ω is uncountably infinite. So in a disjoint sequence in $\mathcal{F}_0$ at most one member can be co-finite, and Case 3 is vacuous.

Therefore Cases 1 and 2 are exhaustive and P is σ-additive on $\mathcal{F}_0$. Recall for Case 1 that σ-additivity of a set function on a field is only asserted for disjoint sequences whose union again lies in that field: if infinitely many of the A_n were distinct, non-empty and finite, then $\cup_{n\geqslant 1} A_n$ would be countably infinite, and in an uncountable Ω such a set is neither finite nor co-finite, so it would not lie in $\mathcal{F}_0$ and there would be nothing to verify.

5 Homework 5

HW5 (due on 10/02): 2.6 - 3, 6, 7.

Problem 2.6.3

Let $(\Omega, \mathscr{B}, \mathbf{P})$ be a probability space. Show for events $B_l \subset A_l$, the following generalization of subadditivity

$$\mathbf{P}(\cup_l A_l) - \mathbf{P}(\cup_l B_l) \leqslant \sum_l (\mathbf{P}(A_l) - \mathbf{P}(B_l))$$

Solution: Because $B_l \subset A_l$ for all l, we have $\cup_l B_l \subseteq \cup_l A_l$, and moreover

$$\bigcup_l A_l \backslash \bigcup_l B_l \subseteq \bigcup_l (A_l \backslash B_l)$$

Indeed, if x belongs to some A_{l_0} but to no B_l, then in particular $x \notin B_{l_0}$, so $x \in A_{l_0} \backslash B_{l_0}$.
Thus,

$$\begin{aligned}\mathbf{P}(\cup_l A_l) - \mathbf{P}(\cup_l B_l) &= \mathbf{P}\big(\cup_l A_l \backslash \cup_l B_l\big) \leqslant \mathbf{P}(\cup_l (A_l \backslash B_l)) \\ &\leqslant \sum_l \mathbf{P}(A_l \backslash B_l) = \sum_l (\mathbf{P}(A_l) - \mathbf{P}(B_l))\end{aligned}$$

Problem 2.6.6

Say events $A_1, A_2, \ldots$ are almost disjoint if

$$\mathbf{P}(A_i \cap A_j) = 0, \qquad i \neq j$$

Show for such events

$$\mathbf{P}(\cup_{j \geqslant 1} A_j) = \sum_{j \geqslant 1} \mathbf{P}(A_j)$$

Solution: See that for finite case, we have

$$\mathbf{P}(A_i \cup A_j) = \mathbf{P}(A_i) + \mathbf{P}(A_j) - \mathbf{P}(A_i \cap A_j) = \mathbf{P}(A_i) + \mathbf{P}(A_j)$$

Thus, for every $n \in \mathbb{Z}_{\geqslant 1}$, we have

$$\mathbf{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mathbf{P}(A_i)$$

See that $B_n := \cup_{i=1}^n A_i$ is an increasing sequence of events, and $B_n \uparrow \cup_{n=1}^\infty A_n =: A$ as $n \to \infty$. Then by continuity of probability measure, we have

$$\mathbf{P}(A) = \lim_{n \to \infty} \mathbf{P}(B_n) = \lim_{n \to \infty} \mathbf{P}(\cup_{i=1}^n A_i) = \lim_{n \to \infty} \sum_{i=1}^n \mathbf{P}(A_i) = \sum_{n=1}^\infty \mathbf{P}(A_n)$$

which completes the proof.

Problem 2.6.7 (Coupon collecting)

Suppose there are N different types of coupons available when buying cereal; each box contains one coupon and the collector is seeking to collect one of each in order to win a prize. After buying n boxes, what is the probability p_n that the collector has at least one of each type? (Consider sampling with replacement from a population of N distinct elements. The sample size is $n > N$. Use inclusion-exclusion formula.)

Solution: Let the random variable X_n^i being the number coupons of type i after buying $n > N$ boxes where $i = 1, \dots, N$. We want to know $p_n := \mathbf{P}(X_n^1 \geqslant 1, \dots, X_n^N \geqslant 1)$. Assume it is equal likely to get each type of box each time, i.e. for all $i = 1, \dots, N$

$$\mathbf{P}(X_1^i = 1) = \frac{1}{N}, \qquad \mathbf{P}(X_1^i = 0) = \frac{N-1}{N}$$

Here X_1^i follows a Bernoulli distribution with probability $\frac{1}{N}$, and X_n^i follows a binomial distribution, i.e. $X_n^i \sim \mathrm{Bin}(n, \frac{1}{N})$. Thus

$$\mathbf{P}(X_n^i = 0) = \left(1 - \frac{1}{N}\right)^n$$

Then consider the joint distribution (X_n^i, X_n^j) for $i < j$. and

$$\begin{aligned}\mathbf{P}(X_n^i = 0, X_n^j = 0) &= \mathbf{P}(X_n^i = 0 | X_n^j = 0)\, \mathbf{P}(X_n^j = 0) \\ &= \left(1 - \frac{1}{N-1}\right)^n \left(1 - \frac{1}{N}\right)^n \\ &= \left(1 - \frac{2}{N}\right)^n\end{aligned}$$

In fact, the general case follows from a direct count. Fix k distinct types $i_1 < \cdots < i_k$. The event $\{X_n^{i_1} = 0, \dots, X_n^{i_k} = 0\}$ means that every one of the n independent draws avoids all k of these types, and a single draw does so with probability $\frac{N-k}{N}$. Hence

$$\mathbf{P}(X_n^{i_1} = 0, \dots, X_n^{i_k} = 0) = \left(1 - \frac{k}{N}\right)^n$$

(the computation for $k = 2$ above serves as a consistency check).

Consider the complement

$$\mathbf{P}(\cup_{k=1}^N \{X_n^k = 0\}) = 1 - \mathbf{P}(X_n^1 \geqslant 1, \dots, X_n^N \geqslant 1)$$

Use inclusion exclusion formula, we have

$$\begin{aligned}\mathbf{P}(\cup_{k=1}^N \{X_n^k = 0\}) = &\sum_{k=1}^N \mathbf{P}(X_n^k = 0) - \sum_{i<j} \mathbf{P}(X_n^i = 0, X_n^j = 0) \\ &+ \sum_{i<j<k} \mathbf{P}(X_n^i = 0, X_n^j = 0, X_n^k = 0) - \dots \\ &\quad + (-1)^{N+1}\, \mathbf{P}(X_n^1 = 0, \dots, X_n^N = 0) \\ = &\, N\left(1 - \frac{1}{N}\right)^n - \binom{N}{2}\left(1 - \frac{2}{N}\right)^n + \dots + (-1)^{N+1}\left(1 - \frac{N}{N}\right)^n \\ = &\sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \left(1 - \frac{k}{N}\right)^n\end{aligned}$$

Thus, combining the complement relation with the inclusion–exclusion computation,

$$p_n = \mathbf{P}(X_n^1 \geqslant 1, \ldots, X_n^N \geqslant 1) = 1 - \mathbf{P}\left(\bigcup_{k=1}^{N} \{X_n^k = 0\}\right) = 1 - \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left(1 - \frac{k}{N}\right)^n$$

which is the required probability.

6 Homework 6

HW6 (due on 10/09): 2.6 - 9 (b) (d), 11, 19(a) (b) (c).

Problem 2.6.9

Background: Call two sets $A_1, A_2 \in \mathscr{B}$ equivalent if $\mathbf{P}(A_1 \triangle A_2) = 0$. For a set $A \in \mathscr{B}$, define the equivalence class

$$A^{\#} := \{B \in \mathscr{B} : \mathbf{P}(A \triangle B) = 0\}$$

This decompose $\mathscr{B}$ into equivalence classes. Write

$$\mathbf{P}^{\#}(A^{\#}) = \mathbf{P}(A), \quad \forall A \in A^{\#}$$

In practice we drop #'s; that is identify the equivalence classes with the members.

An atom in a probability space $(\Omega, \mathscr{B}, \mathbf{P})$ is defined as (the equivalence class of) a set $A \in \mathscr{B}$ such that $\mathbf{P}(A) > 0$, and if $B \subset A$ and $B \in \mathscr{B}$, then $\mathbf{P}(B) = 0$, or $\mathbf{P}(A \backslash B) = 0$. Furthermore the probability space is called non-atomic if there are no atoms; that is, $A \in \mathscr{B}$ and $\mathbf{P}(A) > 0$ imply that there exists a $B \in \mathscr{B}$ such that $B \subset A$ and $0 < \mathbf{P}(B) < \mathbf{P}(A)$.

Remark

When we say an atom $A \in \mathscr{B}$ with $\mathbf{P}(A) > 0$, we mean that for every measurable $B \subset A$, either $\mathbf{P}(B) = 0$ or $\mathbf{P}(A \backslash B) = 0$ (equivalently, $\mathbf{P}(B) \in \{0, \mathbf{P}(A)\}$).

(a)

If $\Omega = \mathbb{R}$, and $\mathbf{P}$ is determined by a distribution function $F(x)$, show that the atoms are $\{x \in \Omega : F(x) - F(x-) > 0\}$.

Solution: Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a random variable $X \sim F$. What this question mean is to consider the push-forward measure of $\mathbf{P}$ by this random variable X, i.e.

$$F(x) := \mathbf{P}((-\infty, x]) := (\mathbf{P} \circ X^{-1})((-\infty, x])$$

Since $\Omega = \mathbb{R}$, we can think X as an identity map. Let $x \in \mathbb{R}$, then

$$\mathbf{P}(\{x\}) = F(x) - F(x-)$$

where $F(x-)$ is the left-hand limit of F at $x \in \mathbb{R}$. Remember that DF F is right-continuous. Now we have two cases,

- F is continuous at x, i.e. $F(x) - F(x-) = 0$. Then $\mathbf{P}(\{x\}) = 0$, which means $\{x\}$ is not an atom.
- F has a jump at x, i.e. $F(x) - F(x-) > 0$. Then $\mathbf{P}(\{x\}) > 0$. So $\{x\}$ can be an atom

For the second case, it is obvious that if $B \subseteq \{x\}$, either $B = \{x\}$ or $B = \varnothing$. Then $\mathbf{P}(B) = \mathbf{P}(\{x\})$ or $\mathbf{P}(B) = 0$. Hence, $\{x\}$ is an atom.

It remains to show the converse: every atom of $\mathbf{P}$ is equivalent to such a singleton. Let $A \in \mathscr{B}(\mathbb{R})$ be an atom and write $c := \mathbf{P}(A) > 0$. For each $n \in \mathbb{Z}_{\geqslant 1}$, the dyadic intervals $I_{n,j} := \left(\frac{j}{2^n}, \frac{j+1}{2^n}\right]$, $j \in \mathbb{Z}$, partition $\mathbb{R}$, so

$$c = \mathbf{P}(A) = \sum_{j \in \mathbb{Z}} \mathbf{P}(A \cap I_{n,j})$$

Since A is an atom, each term is either 0 or c, and since the terms sum to c, exactly one index j_n has $\mathbf{P}(A \cap I_{n,j_n}) = c$. Write $I_n := I_{n,j_n}$. The level-$(n+1)$ dyadic intervals refine the level-n ones, and the unique level-$(n+1)$ interval carrying mass c must lie inside I_n, because the level-$(n+1)$ intervals contained in I_n already account for all of $\mathbf{P}(A \cap I_n) = c$. Hence the I_n are nested, with lengths $2^{-n} \to 0$. By continuity from above,

$$\mathbf{P}\left(A \cap \bigcap_{n \geqslant 1} I_n\right) = \lim_{n \to \infty} \mathbf{P}(A \cap I_n) = c > 0$$

so $\bigcap_{n \geqslant 1} I_n$ is non-empty; having diameter 0, it is a single point $\{x\}$. Then $\mathbf{P}(\{x\}) \geqslant \mathbf{P}(A \cap \{x\}) = c > 0$, so $F(x) - F(x-) = \mathbf{P}(\{x\}) > 0$, and

$$\mathbf{P}(A \backslash \{x\}) = \mathbf{P}(A) - \mathbf{P}(A \cap \{x\}) = c - c = 0$$

so A is equivalent to $\{x\}$ in the sense of the background above. Hence, up to equivalence, the atoms of $\mathbf{P}$ are exactly the singletons in $\{x \in \Omega : F(x) - F(x-) > 0\}$.

(b)

If $(\Omega, \mathscr{B}, \mathbf{P}) = ((0,1], \mathscr{B}((0,1]), \lambda)$, where λ is Lebesgue measure, then the probability space is non-atomic.

Solution: Suppose this probability space is atomic. Say $A \in \mathscr{B}((0,1])$ is such an atom. Then we have $\lambda(A) > 0$ and for every measurable $B \subset A$, either $\lambda(B) = 0$ or $\lambda(B) = \lambda(A)$.

Define $f(t) := \lambda(A \cap (0,t])$ for $t \in [0,1]$. Then $f(0) = 0$, $f(1) = \lambda(A) > 0$, and f is continuous, because for $s < t$

$$|f(t) - f(s)| = \lambda(A \cap (s,t]) \leqslant \lambda((s,t]) = t - s$$

By the intermediate value theorem there exists $t^* \in (0,1)$ with $f(t^*) = \frac{\lambda(A)}{2}$. Set $B := A \cap (0, t^*] \in \mathscr{B}((0,1])$. Then $B \subset A$ and

$$0 < \lambda(B) = \frac{\lambda(A)}{2} < \lambda(A)$$

which contradicts the defining property of the atom A: every measurable subset of A must have measure 0 or $\lambda(A)$. Hence, $((0,1], \mathscr{B}((0,1]), \lambda)$ is non-atomic.

(c)

Show that two distinct atoms have intersection which is the empty set. (The sets A, B are distinct means $\mathbf{P}(A \triangle B) > 0$. The exercise then requires showing $\mathbf{P}(AB \triangle \varnothing) = 0$.)
Solution: Let $A, B \in \mathscr{B}$ be two atoms of the probability space. Assume $\mathbf{P}(A \triangle B) > 0$. Want to show $\mathbf{P}(A \cap B) = 0$. Because $A \cap B \subset A, B$, by definition of atom, we have either $\mathbf{P}(A \cap B) = 0$ or $\mathbf{P}(A \cap B) = \mathbf{P}(A)$ together with $\mathbf{P}(A \cap B) = \mathbf{P}(B)$.

Let's say we don't have $\mathbf{P}(A \cap B) = 0$. Then we have $\mathbf{P}(A \cap B) = \mathbf{P}(A)$. Then

$$\mathbf{P}(A \backslash B) = \mathbf{P}(A) - \mathbf{P}(A \cap B) = 0$$

and

$$\mathbf{P}(B \backslash A) = \mathbf{P}(B) - \mathbf{P}(A \cap B) = 0$$

But this means

$$\mathbf{P}(A \triangle B) = \mathbf{P}(A \backslash B) + \mathbf{P}(B \backslash A) = 0$$

which contradicts the assumption $\mathbf{P}(A \triangle B) > 0$.

(d)

A probability space contains at most countably many atoms. (Hint: What is the maximum number of atoms that the space can contain that have probability at least $1/n$?
Solution: Let $(\Omega, \mathscr{B}, \mathbf{P})$ be a probability space, and suppose it contains atoms.

From the hint, consider the collection of sets

$$\mathcal{S}_n := \left\{ A \in \mathscr{B} : \mathbf{P}(A) \geqslant \frac{1}{n} \right\}$$

Consider atoms of the probability space in $\mathcal{S}_n$. Because each atom $A \in \mathscr{B}$ can only have probability $\mathbf{P}(A) \geqslant \frac{1}{n}$, and use the result from Part (c) that two distinct atoms can only have empty intersection. There can be only at most n distinct atoms of $\mathscr{B}$ sitting inside $\mathcal{S}_n$. Otherwise, if we had $n+1$ or more distinct atoms $A_i, i \geqslant 1$, then

$$\mathbf{P}\left(\bigcup_{i=1}^{n+1} A_i \right) = \sum_{i=1}^{n+1} \mathbf{P}(A_i) \geqslant \sum_{i=1}^{n+1} \frac{1}{n} = \frac{n+1}{n} > 1$$

(the A_i are pairwise almost disjoint by Part (c), so additivity holds as in Problem 2.6.6), which contradicts

$$\mathbf{P}\left(\bigcup_{i=1}^{n+1} A_i \right) \leqslant \mathbf{P}(\Omega) = 1$$

using monotonicity of probability measure $\mathbf{P}$ and $\cup_{i=1}^{n+1} A_i \subseteq \Omega$. Furthermore, since every atom A of the space satisfies $\mathbf{P}(A) > 0$ by definition, the Archimedean property of $\mathbb{R}$ supplies an $n \in \mathbb{Z}_{\geqslant 1}$ with $\mathbf{P}(A) \geqslant \frac{1}{n}$, so that $A \in \mathcal{S}_n$. Hence every atom of $(\Omega, \mathscr{B}, \mathbf{P})$ lies in at least one of the collections $\mathcal{S}_n$, and by the previous paragraph each $\mathcal{S}_n$ contains at most n distinct atoms.

Define

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n = \bigcup_{n=1}^{\infty} \left\{ A \in \mathscr{B} : \mathbf{P}(A) \geqslant \frac{1}{n} \right\}$$

Because each $\mathcal{S}_n$ has only n atoms, thus $\mathcal{S}$ has countable union of finitely many atoms, which will have countably many atoms.

Overall, we proved the probability space $(\Omega, \mathscr{B}, \mathbf{P})$ can only contain countably many atoms.

(e)

If a probability space $(\Omega, \mathscr{B}, \mathbf{P})$ contains no atoms, then for every $a \in (0, 1]$ there exists at least one set $A \in \mathscr{B}$ such that $\mathbf{P}(A) = a$. (One way of doing this uses Zorn's lemma.)

Solution: Throughout we use "non-atomic" in the form stated in the background above: if $A \in \mathscr{B}$ and $\mathbf{P}(A) > 0$, then there is a $B \in \mathscr{B}$ with $B \subset A$ and $0 < \mathbf{P}(B) < \mathbf{P}(A)$.

Lemma (sets of arbitrarily small positive probability). Let $(\Omega, \mathscr{B}, \mathbf{P})$ be non-atomic, $A \in \mathscr{B}$ with $\mathbf{P}(A) > 0$, and $\delta > 0$. Then there is a $D \in \mathscr{B}$ with $D \subseteq A$ and $0 < \mathbf{P}(D) < \delta$.

Proof. First we halve. Let $C \in \mathscr{B}$ with $\mathbf{P}(C) > 0$. By non-atomicity there is $B \subseteq C$ with $0 < \mathbf{P}(B) < \mathbf{P}(C)$, and then

$$0 < \mathbf{P}(C) - \mathbf{P}(B) = \mathbf{P}(C \backslash B) < \mathbf{P}(C)$$

so B and $C \backslash B$ are both measurable subsets of C of positive probability. Since $\mathbf{P}(B) + \mathbf{P}(C \backslash B) = \mathbf{P}(C)$, at least one of them has probability at most $\frac{1}{2}\mathbf{P}(C)$; call it $h(C)$. Thus

$$h(C) \subseteq C, \qquad \mathbf{P}(h(C)) > 0, \qquad \mathbf{P}(h(C)) \leqslant \tfrac{1}{2}\mathbf{P}(C)$$

Now set $A_0 := A$ and $A_{k+1} := h(A_k)$, which is legitimate at every stage because $\mathbf{P}(A_k) > 0$. Then $A_0 \supseteq A_1 \supseteq \ldots$, each $\mathbf{P}(A_k) > 0$, and by induction

$$\mathbf{P}(A_k) \leqslant \frac{1}{2^k}\mathbf{P}(A) \longrightarrow 0 \qquad \text{as } k \to \infty$$

So $\mathbf{P}(A_k) < \delta$ for k large enough, and $D := A_k$ has the required properties. This proves the lemma.

Construction of the set of probability a. Fix $a \in (0, 1]$. We build an increasing sequence $\varnothing = B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots$ in $\mathscr{B}$ with $\mathbf{P}(B_n) \leqslant a$ for every n. Suppose B_n is already constructed. Put

$$s_n := \sup\{\mathbf{P}(C) : C \in \mathscr{B},\ C \subseteq \Omega \backslash B_n,\ \mathbf{P}(B_n) + \mathbf{P}(C) \leqslant a\}$$

The collection on the right contains $C = \varnothing$, so it is non-empty and $0 \leqslant s_n \leqslant 1$. Choose $C_n \in \mathscr{B}$ with

$$C_n \subseteq \Omega \backslash B_n, \qquad \mathbf{P}(B_n) + \mathbf{P}(C_n) \leqslant a, \qquad \mathbf{P}(C_n) \geqslant \frac{s_n}{2}$$

which is possible by the definition of a supremum when $s_n > 0$, and by taking $C_n = \varnothing$ when $s_n = 0$. Set $B_{n+1} := B_n \cup C_n$. Since C_n is disjoint from B_n,

$$\mathbf{P}(B_{n+1}) = \mathbf{P}(B_n) + \mathbf{P}(C_n) \leqslant a$$

as required.

The limit set has probability exactly a. The sets $C_0, C_1, \ldots$ are pairwise disjoint: $C_n \subseteq \Omega \backslash B_n$, while $C_m \subseteq B_{m+1} \subseteq B_n$ for every $m < n$. Let

$$B := \bigcup_{n \geqslant 0} B_n = \bigcup_{n \geqslant 0} C_n \in \mathscr{B}$$

By continuity from below, $\mathbf{P}(B) = \lim_{n\to\infty} \mathbf{P}(B_n) \leqslant a$, and by σ-additivity over the disjoint C_n,

$$\sum_{n \geqslant 0} \mathbf{P}(C_n) = \mathbf{P}(B) \leqslant a \leqslant 1 < \infty \implies \mathbf{P}(C_n) \longrightarrow 0 \quad \text{ as } n \to \infty$$

Suppose for contradiction that $\mathbf{P}(B) < a$, and put $\delta := a - \mathbf{P}(B) > 0$. From $\mathbf{P}(B) < a \leqslant 1$ we get $\mathbf{P}(\Omega \backslash B) = 1 - \mathbf{P}(B) > 0$, so the lemma applied to $\Omega \backslash B$ gives a $D \in \mathscr{B}$ with

$$D \subseteq \Omega \backslash B, \qquad 0 < \mathbf{P}(D) < \delta$$

For every n we then have $D \subseteq \Omega \backslash B \subseteq \Omega \backslash B_n$ and

$$\mathbf{P}(B_n) + \mathbf{P}(D) \leqslant \mathbf{P}(B) + \mathbf{P}(D) < \mathbf{P}(B) + \delta = a$$

so D is one of the competitors in the supremum defining s_n. Hence $s_n \geqslant \mathbf{P}(D)$ for every n, and therefore

$$\mathbf{P}(C_n) \geqslant \frac{s_n}{2} \geqslant \frac{\mathbf{P}(D)}{2} > 0 \qquad \text{for every } n \geqslant 0$$

which contradicts $\mathbf{P}(C_n) \to 0$. Consequently $\mathbf{P}(B) = a$, and $A := B$ is a set as required.

Remark The hint suggests Zorn's lemma, but the exhaustion argument above uses only countable additivity together with countably many choices, so no maximal-element argument is needed.

(f)

For every probability space $(\Omega, \mathscr{B}, \mathbf{P})$ and any $\epsilon > 0$, there exists a finite partition of Ω by $\mathscr{B}$ sets, each of whose elements either has probability $\leqslant \epsilon$ or is an atom with probability $> \epsilon$.

Solution: Fix $\epsilon > 0$. The idea is to peel off the finitely many "large" atoms, lump the remaining atoms into one small set, and cut the non-atomic remainder into slices of size ϵ using Part (e).

Step 1: a set equivalent to an atom is an atom Let A be an atom and let $A' \in \mathscr{B}$ with $\mathbf{P}(A \triangle A') = 0$. Then $\mathbf{P}(A') = \mathbf{P}(A) > 0$. Let $B \in \mathscr{B}$ with $B \subseteq A'$. Since $B \triangle (B \cap A) = B \backslash A \subseteq A' \backslash A$, we get $\mathbf{P}(B) = \mathbf{P}(B \cap A)$. As $B \cap A \subseteq A$ and A is an atom, either $\mathbf{P}(B \cap A) = 0$, and then $\mathbf{P}(B) = 0$, or $\mathbf{P}(A \backslash B) = 0$, and then

$$\mathbf{P}(A' \backslash B) \leqslant \mathbf{P}(A' \backslash A) + \mathbf{P}(A \backslash B) = 0$$

So A' is an atom with $\mathbf{P}(A') = \mathbf{P}(A)$. This is just the statement that being an atom is a property of the equivalence class, which is the convention announced in the background of this problem.

Step 2: the atoms can be listed as a disjoint sequence By Part (d) there are at most countably many atoms. Let $A_1, A_2, \ldots$ be representatives of the distinct ones (a finite or an infinite list). By Part (c), $\mathbf{P}(A_i \cap A_j) = 0$ for $i \neq j$. Put

$$G_j := A_j \backslash \bigcup_{i<j} A_i \in \mathscr{B}$$

Then $G_1, G_2, \ldots$ are pairwise disjoint, and

$$\mathbf{P}(A_j \triangle G_j) = \mathbf{P}\left(A_j \cap \bigcup_{i<j} A_i \right) \leqslant \sum_{i<j} \mathbf{P}(A_j \cap A_i) = 0$$

so by Step 1 each G_j is an atom with $\mathbf{P}(G_j) = \mathbf{P}(A_j)$, and every atom of the space is equivalent to one of the G_j.

Step 3: what is left over carries no atom Put

$$N := \Omega \backslash \bigcup_j G_j \in \mathscr{B}$$

Suppose $C \in \mathscr{B}$, $C \subseteq N$, and C is an atom. By Step 2, C is equivalent to some G_j, so $\mathbf{P}(C) = \mathbf{P}(C \cap G_j)$; but $C \subseteq N$ is disjoint from G_j, so $\mathbf{P}(C) = 0$, contradicting $\mathbf{P}(C) > 0$. Hence no measurable subset of N is an atom of $\mathbf{P}$. If $\mathbf{P}(N) > 0$, then in the probability space

$$\left(N,\ \{B \in \mathscr{B} : B \subseteq N\},\ \frac{\mathbf{P}(\cdot)}{\mathbf{P}(N)} \right)$$

a set $C \subseteq N$ is an atom for the normalised measure exactly when it is an atom for $\mathbf{P}$, since dividing by the constant $\mathbf{P}(N) > 0$ changes neither "$\mathbf{P}(\cdot) = 0$" nor "$\mathbf{P}(C \backslash B) = 0$". So this space is non-atomic, and the same is true of any measurable subset of N of positive probability.

Step 4: only finitely many atoms are kept separate The G_j are disjoint, so $\sum_j \mathbf{P}(G_j) \leqslant 1 < \infty$. Choose J so large that

$$\sum_{j>J} \mathbf{P}(G_j) < \epsilon$$

(if the list of atoms is finite, take J to be its length and read the tail as $\varnothing$), and put

$$E := \bigcup_{j>J} G_j, \qquad \mathbf{P}(E) < \epsilon$$

No extra care is needed to catch the atoms of large probability: any G_j with $j > J$ satisfies $\mathbf{P}(G_j) \leqslant \mathbf{P}(E) < \epsilon$, so every atom of probability $> \epsilon$ is automatically among $G_1, \ldots, G_J$.

Step 5: slicing the non-atomic part If $\mathbf{P}(N) = 0$, set $M := 1$ and $H_1 := N$; then $\mathbf{P}(H_1) = 0 \leqslant \epsilon$. Otherwise let

$$M := \left\lceil \frac{\mathbf{P}(N)}{\epsilon} \right\rceil$$

and construct $H_1, \ldots, H_M$ recursively. Suppose disjoint $H_1, \ldots, H_{i-1} \subseteq N$ with $\mathbf{P}(H_1) = \ldots = \mathbf{P}(H_{i-1}) = \epsilon$ have been chosen, where $i \leqslant M - 1$, and put

$$N_i := N \backslash (H_1 \cup \ldots \cup H_{i-1}), \qquad \mathbf{P}(N_i) = \mathbf{P}(N) - (i-1)\epsilon$$

From $M - 1 < \mathbf{P}(N)/\epsilon$ and $i - 1 \leqslant M - 2$ we get $\mathbf{P}(N_i) > \epsilon$, so

$$\frac{\epsilon}{\mathbf{P}(N_i)} \in (0, 1]$$

and Part (e), applied to the non-atomic probability space carried by N_i (Step 3), yields $H_i \subseteq N_i$ with $\mathbf{P}(H_i) = \epsilon$. Finally take

$$H_M := N \backslash (H_1 \cup \ldots \cup H_{M-1}), \qquad \mathbf{P}(H_M) = \mathbf{P}(N) - (M-1)\epsilon \leqslant \epsilon$$

the last inequality because $M\epsilon \geqslant \mathbf{P}(N)$. Thus $H_1, \ldots, H_M$ is a finite partition of N into $\mathscr{B}$ sets each of probability at most ϵ.

Conclusion The finite family

$$\{G_1, \ldots, G_J\} \cup \{E\} \cup \{H_1, \ldots, H_M\}$$

consists of pairwise disjoint $\mathscr{B}$ sets whose union is Ω, so it is a finite partition of Ω by $\mathscr{B}$ sets (discard the empty members if one insists that the blocks be non-empty). Every block is of one of the two permitted kinds:

- each G_j with $j \leqslant J$ is an atom, so either $\mathbf{P}(G_j) \leqslant \epsilon$, or $\mathbf{P}(G_j) > \epsilon$ and it is an atom of probability $> \epsilon$;
- $\mathbf{P}(E) < \epsilon$;
- $\mathbf{P}(H_i) \leqslant \epsilon$ for $1 \leqslant i \leqslant M$.

This completes the proof.

(g) (Metric space)

On the set of equivalence classes, define

$$d(A_1^\#, A_2^\#) = \mathbf{P}(A_1 \triangle A_2)$$

where $A_i \in A_i^\#$ for $i = 1, 2$. Show d is a metric on the set of equivalence classes. Verify

$$|\,\mathbf{P}(A_1) - \mathbf{P}(A_2)| \leqslant \mathbf{P}(A_1 \triangle A_2)$$

so that $\mathbf{P}^\#$ is uniformly continuous on the set of equivalence classes. $\mathbf{P}$ is a σ-additive is equivalent to

$$\mathscr{B} \ni A_n \downarrow \varnothing \quad \text{implies} \quad d(A_n^\#, \varnothing^\#) \longrightarrow 0$$

Solution: The only hard part of showing d is a metric is triangular inequality. Consider $A_1^\#, A_2^\#, A_3^\#$. See that

$$A_1 \triangle A_3 \subset (A_1 \triangle A_2) \cup (A_2 \triangle A_3)$$

Hence by monotonicity and sub-additivity of probability measure, we have

$$\begin{aligned} d(A_1^\#, A_3^\#) &= \mathbf{P}(A_1 \triangle A_3) \\ &\leqslant \mathbf{P}((A_1 \triangle A_2) \cup (A_2 \triangle A_3)) \\ &\leqslant \mathbf{P}(A_1 \triangle A_2) + \mathbf{P}((A_2 \triangle A_3)) \\ &= d(A_1^\#, A_2^\#) + d(A_2^\#, A_3^\#) \end{aligned}$$

For any $A_1, A_2 \in \mathscr{B}$, we have

$$\begin{aligned} \mathbf{P}(A_1) &= \mathbf{P}(A_1 \cap A_2) + \mathbf{P}(A_1 \backslash A_2) \\ \mathbf{P}(A_2) &= \mathbf{P}(A_1 \cap A_2) + \mathbf{P}(A_2 \backslash A_1) \end{aligned}$$

Hence, by triangular inequality of absolute value $|\cdot|$. we have

$$|\mathbf{P}(A_1) - \mathbf{P}(A_2)| = |\mathbf{P}(A_1 \backslash A_2) - \mathbf{P}(A_2 \backslash A_1)| \leqslant \mathbf{P}(A_1 \backslash A_2) + \mathbf{P}(A_2 \backslash A_1) = \mathbf{P}(A_1 \triangle A_2)$$

Suppose $\mathscr{B} \ni A_n \downarrow \varnothing$. Then by monotone property of probability measure, we have $\mathbf{P}(A_n) \downarrow 0$ as $n \to \infty$. Observe that

$$d(A_n^\#, \varnothing^\#) = \mathbf{P}(A_n \triangle \varnothing) = \mathbf{P}(A_n) \downarrow 0 \quad \text{as } n \to \infty$$

This completes the proof of this part.

Problem 2.6.11

Suppose $\{B_n : n \geqslant 1\}$ are events with $\mathbf{P}(B_n) = 1$ for all n. Show

$$\mathbf{P}\Big(\bigcap_{n \geqslant 1} B_n\Big) = 1$$

Solution: We show

$$\mathbf{P}\Bigg(\bigcap_{n\geqslant 1} B_n\Bigg) \geqslant 1 \qquad \text{and} \qquad \mathbf{P}\Bigg(\bigcap_{n\geqslant 1} B_n\Bigg) \leqslant 1$$

It is clear that

$$\mathbf{P}\Bigg(\bigcap_{n\geqslant 1} B_n\Bigg) \leqslant \mathbf{P}(B_k) = 1 \quad \text{for every } k \geqslant 1$$

Also,

$$\begin{aligned}
\mathbf{P}\Bigg(\bigcup_{n\geqslant 1} B_n^c\Bigg) &\leqslant \sum_{n\geqslant 1} \mathbf{P}(B_n^c) = \sum_{n\geqslant 1}(1-\mathbf{P}(B_n)) = \sum_{n\geqslant 1} 0 = 0 \\
-\mathbf{P}\Bigg(\bigcup_{n\geqslant 1} B_n^c\Bigg) &\geqslant 0 \\
1-\mathbf{P}\Bigg(\bigcup_{n\geqslant 1} B_n^c\Bigg) &\geqslant 1 \\
\mathbf{P}\Bigg(\bigcap_{n\geqslant 1} B_n\Bigg) &\geqslant 1
\end{aligned}$$

This completes the proof.

Problem 2.6.19 (Completion)

Let $(\Omega, \mathscr{B}, \mathbf{P})$ be a probability space. Call a set N $\mathbf{P}$-null if $N \in \mathscr{B}$ and $\mathbf{P}(N) = 0$. Call a set $B \subseteq \Omega$ negligible if there exists a $\mathbf{P}$-null set N such that $B \subseteq N$. Notice that for B to be negligible, it is not required that B be measurable. Denote the set of all negligible subsets by $\mathcal{N}$. Call $\mathscr{B}$ complete (with respect to $\mathbf{P}$) if every negligible set is $\mathbf{P}$-null.

What if $\mathscr{B}$ is not complete? Define

$$\mathscr{B}^* := \{A \cup M : A \in \mathscr{B}, M \in \mathcal{N}\}$$

(a)

Show that $\mathscr{B}^*$ is a σ-field.

Solution: Check 3 axioms of σ-field.

Since empty set $\varnothing$ has $\mathbf{P}(\varnothing) = 0$ which is clearly a $\mathbf{P}$-null set. Hence $\varnothing \in \mathcal{N}$. Also, $\varnothing \in \mathscr{B}$ because $\mathscr{B}$ is a σ-field. Choose $A = \varnothing \in \mathscr{B}$ and $M = \varnothing \in \mathcal{N}$. Thus, $\varnothing = \varnothing \cup \varnothing \in \mathscr{B}^*$.

Let $A \cup M \in \mathscr{B}^*$ where $A \in \mathscr{B}$ and $M \in \mathcal{N}$. For the negligible set $M \in \mathcal{N}$, there exists a $\mathbf{P}$-null set $N \in \mathscr{B}$ such that $M \subseteq N$. Hence, taking complement, we have $M^c \supseteq N^c$. Consider

$$\begin{aligned}
(A \cup M)^c &= A^c \cap M^c \\
&= A^c \cap (N^c \cup (N\backslash M)) \\
&= (A^c \cap N^c) \cup (A^c \cap (N\backslash M))
\end{aligned}$$

Because $A, N \in \mathscr{B}$, then $A^c \cap N^c \in \mathscr{B}$. See that $A^c \cap (N \backslash M) \subseteq N$ where N is a $\mathbf{P}$-null set. Hence, $A^c \cap (N \backslash M) \in \mathcal{N}$ is a negligible set. Overall, $(A \cup M)^c \in \mathscr{B}^*$.

Let $A_i \cup M_i \in \mathscr{B}^*$ for $i \geqslant 1$. For each M_i, there exists a $\mathbf{P}$-null set $N_i \in \mathscr{B}$ such that $M_i \subseteq N_i$. Then $\cup_{i \geqslant 1} M_i \subseteq \cup_{i \geqslant 1} N_i$. Observe that $\cup_{i \geqslant 1} N_i$ is also a null set by sub-additivity of probability measure

$$\mathbf{P}(\cup_{i \geqslant 1} N_i) \leqslant \sum_{i \geqslant 1} \mathbf{P}(N_i) = \sum_{i \geqslant 1} 0 = 0$$

Hence, $\cup_{i \geqslant 1} M_i \in \mathcal{N}$ is also a negligible set. Consider

$$\cup_{i \geqslant 1}(A_i \cup M_i) = \big(\cup_{i \geqslant 1} A_i\big) \cup \big(\cup_{i \geqslant 1} M_i\big)$$

where $\cup_{i \geqslant 1} A_i \in \mathscr{B}$ and $\cup_{i \geqslant 1} M_i \in \mathcal{N}$. Hence, $\cup_{i \geqslant 1} A_i \cup M_i \in \mathscr{B}^*$.

This completes the proof of $\mathscr{B}^*$ being a σ-field.

(b)

If $A_i \in \mathscr{B}$ and $M_i \in \mathcal{N}$ for $i = 1, 2$ and

$$A_1 \cup M_1 = A_2 \cup M_2$$

then $\mathbf{P}(A_1) = \mathbf{P}(A_2)$.

Solution: To show $\mathbf{P}(A_1) = \mathbf{P}(A_2)$, we show

$$\mathbf{P}(A_1) \leqslant \mathbf{P}(A_2) \quad \text{and} \quad \mathbf{P}(A_1) \geqslant \mathbf{P}(A_2)$$

Since $M_i \in \mathcal{N}$, there exists $\mathbf{P}$-null sets $N_i \in \mathscr{B}$ such that $M_i \subseteq N_i$ for each $i = 1, 2$. Thus we will have the following inclusion relationship

$$A_1 \cup M_1 = A_2 \cup M_2 \subseteq A_2 \cup N_2, \qquad A_2 \cup M_2 = A_1 \cup M_1 \subseteq A_1 \cup N_1$$

Now, by monotonicity and finite sub-additivity of measure

$$\mathbf{P}(A_1) \leqslant \mathbf{P}(A_2 \cup N_2) \leqslant \mathbf{P}(A_2) + \mathbf{P}(N_2) = \mathbf{P}(A_2) + 0 = \mathbf{P}(A_2)$$

By symmetry, we have

$$\mathbf{P}(A_2) \leqslant \mathbf{P}(A_1 \cup N_1) \leqslant \mathbf{P}(A_1) + \mathbf{P}(N_1) = \mathbf{P}(A_1) + 0 = \mathbf{P}(A_1)$$

Hence, we conclude

$$\mathbf{P}(A_1) = \mathbf{P}(A_2)$$

(c)

Define $\mathbf{P}^* : \mathscr{B}^* \longrightarrow [0, 1]$ by

$$\mathbf{P}^*(A \cup M) = \mathbf{P}(A), \quad A \in \mathscr{B}, M \in \mathcal{N}$$

Show $\mathbf{P}^*$ is an extension of $\mathbf{P}$ to $\mathscr{B}^*$.

Solution: We need to first check if $\mathbf{P}^*$ well-defined, and then check is indeed a probability measure on $\mathscr{B}^*$ where $\mathbf{P}^*|_{\mathscr{B}} = \mathbf{P}$.

Well-definedness of $\mathbf{P}^*$. Let $A_1 \cup M_1, A_2 \cup M_2 \in \mathscr{B}^*$ are sets such that

$$A_1 \cup M_1 = A_2 \cup M_2$$

Then from Part (b), we already know $\mathbf{P}(A_1) = \mathbf{P}(A_2)$. Thus,

$$\mathbf{P}^*(A_1 \cup M_1) = \mathbf{P}(A_1) = \mathbf{P}(A_2) = \mathbf{P}^*(A_2 \cup M_2)$$

Hence, this set function $\mathbf{P}^*$ is well-defined.

Extension $\mathbf{P}^*|_{\mathscr{B}} = \mathbf{P}$. This is pretty clear. For $A \cup M \in \mathscr{B}^*$, choose $M = \varnothing$, so

$$\mathbf{P}^*(A) = \mathbf{P}^*(A \cup \varnothing) = \mathbf{P}(A)$$

$\mathbf{P}^*$ is a probability measure on $\mathscr{B}^*$. It is clear that $\mathbf{P}^*(\varnothing) = \mathbf{P}(\varnothing) = 0$.

Also, for every $A \in \mathscr{B}, M \in \mathcal{N}$ we will have $\mathbf{P}^*(A \cup M) = \mathbf{P}(A) \geqslant 0$ from the fact that $\mathbf{P}$ is a probability measure on $\mathscr{B}$.

The only non-trivial part is σ-additivity. Consider $A_i \cup M_i \in \mathscr{B}^*$ be disjoint sets for $i \geqslant 1$. Note that the A_i are then pairwise disjoint as well: $A_i \cap A_j \subseteq (A_i \cup M_i) \cap (A_j \cup M_j) = \varnothing$ for $i \neq j$.

$$\begin{aligned}
\mathbf{P}^*\left(\bigcup_{i\geqslant 1}(A_i \cup M_i)\right) &= \mathbf{P}^*\left((\bigcup_{i\geqslant 1} A_i) \cup \left(\cup_{i\geqslant 1} M_i\right)\right) \\
&= \mathbf{P}\left(\bigcup_{i\geqslant 1} A_i\right) = \sum_{i\geqslant 1} \mathbf{P}(A_i) \\
&= \sum_{i\geqslant 1} \mathbf{P}^*\big(A_i \cup M_i\big)
\end{aligned}$$

This completes the proof of $\mathbf{P}^*$ being a probability measure.

(d)

If $B \subseteq \Omega$ and $A_i \in \mathscr{B}, i = 1, 2$ and $A_1 \subseteq B \subseteq A_2$ and $\mathbf{P}(A_2 \backslash A_1) = 0$, then show $B \in \mathscr{B}^*$.

Solution: We want to show there exists $A \in \mathscr{B}$ and $N \in \mathcal{N}$ such that $B = A \cup N$. Observe that

$$B = A_1 \cup (B \backslash A_1)$$

See that

$$B \backslash A_1 = B \cap A_1^c \subseteq A_2 \cap A_1^c = A_2 \backslash A_1$$

where $A_2 \backslash A_1 \in \mathscr{B}$ has $\mathbf{P}(A_2 \backslash A_1) = 0$. Thus, $B \backslash A_1 \in \mathcal{N}$. Hence, $B \in \mathscr{B}^*$.

(e)

Show $\mathscr{B}^*$ is complete with respect to $\mathbf{P}^*$. Thus every σ-field has a completion.
Solution: Let $A \cup M \in \mathscr{B}^*$ where $A \in \mathscr{B}$ and $M \in \mathcal{N}$ be such that $\mathbf{P}^*(A \cup M) = 0$. We want to show every negligible set of $A \cup M$ is in $\mathscr{B}^*$, i.e. for any $E \subseteq A \cup M$, we have $E \in \mathscr{B}^*$, i.e. E is a $\mathscr{B}^*$-measurable set.

Firstly, see that

$$\mathbf{P}(A) = \mathbf{P}^*(A \cup M) = 0$$

by assumption.

Since $M \in \mathcal{N}$, there exists a $\mathbf{P}$-null set $N \in \mathscr{B}$ such that $M \subseteq N$. Hence, we have

$$E \subseteq A \cup M \subseteq A \cup N$$

where $A \cup N \in \mathscr{B}$. Because

$$\mathbf{P}(A \cup N) = \underbrace{\mathbf{P}(A)}_{=0} + \underbrace{\mathbf{P}(N)}_{=0} - \underbrace{\mathbf{P}(A \cap N)}_{\leqslant \mathbf{P}(N)=0} = 0$$

Hence, E is also a $\mathbf{P}$-negligible set.

In addition, see that $\varnothing \subseteq E$. And we know $\varnothing \in \mathscr{B}$ by definition of a σ-algebra. Thus, we have the following chain of inclusions

$$\varnothing \subseteq E \subseteq A \cup N$$

Also observe that

$$\mathbf{P}(A \cup N \backslash \varnothing) = \mathbf{P}(A \cup N) = 0$$

Thus, by part (d) of this question, we have $E \in \mathscr{B}^*$. This completes the proof of $\mathscr{B}^*$ being complete.

7 Homework 7

HW 7 (due on 10/30): 2.6 - 19 (d) (e), 27 (a) (b); 3.4 - 2

Problem 2.6.19 (Completion)

Let $(\Omega, \mathscr{B}, \mathbf{P})$ be a probability space. Call a set N $\mathbf{P}$-null if $N \in \mathscr{B}$ and $\mathbf{P}(N) = 0$. Call a set $B \subseteq \Omega$ negligible if there exists a $\mathbf{P}$-null set N such that $B \subseteq N$. Notice that for B to be negligible, it is not required that B be measurable. Denote the set of all negligible subsets by $\mathcal{N}$. Call $\mathscr{B}$ complete (with respect to $\mathbf{P}$) if every negligible set is $\mathbf{P}$-null.

What if $\mathscr{B}$ is not complete? Define

$$\mathscr{B}^* := \{A \cup M : A \in \mathscr{B}, M \in \mathcal{N}\}$$

(a)

Show that $\mathscr{B}^*$ is a σ-field.

Solution: Check 3 axioms of σ-field.

Since empty set $\varnothing$ has $\mathbf{P}(\varnothing) = 0$ which is clearly a $\mathbf{P}$-null set. Hence $\varnothing \in \mathcal{N}$. Also, $\varnothing \in \mathscr{B}$ because $\mathscr{B}$ is a σ-field. Choose $A = \varnothing \in \mathscr{B}$ and $M = \varnothing \in \mathcal{N}$. Thus, $\varnothing = \varnothing \cup \varnothing \in \mathscr{B}^*$.

Let $A \cup M \in \mathscr{B}^*$ where $A \in \mathscr{B}$ and $M \in \mathcal{N}$. For the negligible set $M \in \mathcal{N}$, there exists a $\mathbf{P}$-null set $N \in \mathscr{B}$ such that $M \subseteq N$. Hence, taking complement, we have $M^c \supseteq N^c$. Consider

$$\begin{aligned}(A \cup M)^c &= A^c \cap M^c \\ &= A^c \cap (N^c \cup (N \backslash M)) \\ &= (A^c \cap N^c) \cup (A^c \cap (N \backslash M))\end{aligned}$$

Because $A, N \in \mathscr{B}$, then $A^c \cap N^c \in \mathscr{B}$. See that $A^c \cap (N \backslash M) \subseteq N$ where N is a $\mathbf{P}$-null set. Hence, $A^c \cap (N \backslash M) \in \mathcal{N}$ is a negligible set. Overall, $(A \cup M)^c \in \mathscr{B}^*$.

Let $A_i \cup M_i \in \mathscr{B}^*$ for $i \geqslant 1$. For each M_i, there exists a $\mathbf{P}$-null set $N_i \in \mathscr{B}$ such that $M_i \subseteq N_i$. Then $\cup_{i\geqslant 1} M_i \subseteq \cup_{i \geqslant 1} N_i$. Observe that $\cup_{i\geqslant 1} N_i$ is also a null set by sub-additivity of probability measure

$$\mathbf{P}(\cup_{i\geqslant 1} N_i) \leqslant \sum_{i\geqslant 1} \mathbf{P}(N_i) = \sum_{i \geqslant 1} 0 = 0$$

Hence, $\cup_{i\geqslant 1} M_i \in \mathcal{N}$ is also a negligible set. Consider

$$\cup_{i\geqslant 1}(A_i \cup M_i) = \big(\cup_{i\geqslant 1} A_i\big) \cup \big(\cup_{i\geqslant 1} M_i\big)$$

where $\cup_{i\geqslant 1} A_i \in \mathscr{B}$ and $\cup_{i\geqslant 1} M_i \in \mathcal{N}$. Hence, $\cup_{i\geqslant 1} A_i \cup M_i \in \mathscr{B}^*$.

This completes the proof of $\mathscr{B}^*$ being a σ-field.

(b)

If $A_i \in \mathscr{B}$ and $M_i \in \mathcal{N}$ for $i = 1, 2$ and

$$A_1 \cup M_1 = A_2 \cup M_2$$

then $\mathbf{P}(A_1) = \mathbf{P}(A_2)$.
Solution: To show $\mathbf{P}(A_1) = \mathbf{P}(A_2)$, we show

$$\mathbf{P}(A_1) \leqslant \mathbf{P}(A_2) \quad \text{and} \quad \mathbf{P}(A_1) \geqslant \mathbf{P}(A_2)$$

Since $M_i \in \mathcal{N}$, there exists $\mathbf{P}$-null sets $N_i \in \mathscr{B}$ such that $M_i \subseteq N_i$ for each $i = 1, 2$. Thus we will have the following inclusion relationship

$$A_1 \cup M_1 = A_2 \cup M_2 \subseteq A_2 \cup N_2, \qquad A_2 \cup M_2 = A_1 \cup M_1 \subseteq A_1 \cup N_1$$

Now, by monotonicity and finite sub-additivity of measure

$$\mathbf{P}(A_1) \leqslant \mathbf{P}(A_2 \cup N_2) \leqslant \mathbf{P}(A_2) + \mathbf{P}(N_2) = \mathbf{P}(A_2) + 0 = \mathbf{P}(A_2)$$

By symmetry, we have

$$\mathbf{P}(A_2) \leqslant \mathbf{P}(A_1 \cup N_1) \leqslant \mathbf{P}(A_1) + \mathbf{P}(N_1) = \mathbf{P}(A_1) + 0 = \mathbf{P}(A_1)$$

Hence, we conclude

$$\mathbf{P}(A_1) = \mathbf{P}(A_2)$$

(c)

Define $\mathbf{P}^* : \mathscr{B}^* \longrightarrow [0, 1]$ by

$$\mathbf{P}^*(A \cup M) = \mathbf{P}(A), \quad A \in \mathscr{B}, M \in \mathcal{N}$$

Show $\mathbf{P}^*$ is an extension of $\mathbf{P}$ to $\mathscr{B}^*$.
Solution: We need to first check if $\mathbf{P}^*$ well-defined, and then check is indeed a probability measure on $\mathscr{B}^*$ where $\mathbf{P}^* |_{\mathscr{B}} = \mathbf{P}$.

Well-definedness of $\mathbf{P}^*$. Let $A_1 \cup M_1, A_2 \cup M_2 \in \mathscr{B}^*$ are sets such that

$$A_1 \cup M_1 = A_2 \cup M_2$$

Then from Part (b), we already know $\mathbf{P}(A_1) = \mathbf{P}(A_2)$. Thus,

$$\mathbf{P}^*(A_1 \cup M_1) = \mathbf{P}(A_1) = \mathbf{P}(A_2) = \mathbf{P}^*(A_2 \cup M_2)$$

Hence, this set function $\mathbf{P}^*$ is well-defined.

Extension $\mathbf{P}^* |_{\mathscr{B}} = \mathbf{P}$. This is pretty clear. For $A \cup M \in \mathscr{B}^*$, choose $M = \varnothing$, so

$$\mathbf{P}^*(A) = \mathbf{P}^*(A \cup \varnothing) = \mathbf{P}(A)$$

$\mathbf{P}^*$ is a probability measure on $\mathscr{B}^*$. It is clear that $\mathbf{P}^*(\varnothing) = \mathbf{P}(\varnothing) = 0$.

Also, for every $A \in \mathscr{B}, M \in \mathcal{N}$ we will have $\mathbf{P}^*(A \cup M) = \mathbf{P}(A) \geqslant 0$ from the fact that $\mathbf{P}$ is a probability measure on $\mathscr{B}$.

The only non-trivial part is σ-additivity. Consider $A_i \cup M_i \in \mathscr{B}^*$ be disjoint sets for $i \geqslant 1$. Note that the A_i are then pairwise disjoint as well: $A_i \cap A_j \subseteq (A_i \cup M_i) \cap (A_j \cup M_j) = \varnothing$ for $i \neq j$.

$$\begin{aligned}\mathbf{P}^*\left(\bigcup_{i\geqslant 1}(A_i \cup M_i)\right) &= \mathbf{P}^*\left((\bigcup_{i\geqslant 1} A_i) \cup \left(\cup_{i\geqslant 1} M_i\right)\right)\\ &= \mathbf{P}\left(\bigcup_{i\geqslant 1} A_i\right) = \sum_{i\geqslant 1}\mathbf{P}(A_i)\\ &= \sum_{i\geqslant 1}\mathbf{P}^*\left(A_i \cup M_i\right)\end{aligned}$$

This completes the proof of $\mathbf{P}^*$ being a probability measure.

(d)

If $B \subseteq \Omega$ and $A_i \in \mathscr{B}, i = 1, 2$ and $A_1 \subseteq B \subseteq A_2$ and $\mathbf{P}(A_2 \backslash A_1) = 0$, then show $B \in \mathscr{B}^*$.

Solution: We want to show there exists $A \in \mathscr{B}$ and $N \in \mathcal{N}$ such that $B = A \cup N$. Observe that

$$B = A_1 \cup (B \backslash A_1)$$

See that

$$B \backslash A_1 = B \cap A_1^c \subseteq A_2 \cap A_1^c = A_2 \backslash A_1$$

where $A_2 \backslash A_1 \in \mathscr{B}$ has $\mathbf{P}(A_2 \backslash A_1) = 0$. Thus, $B \backslash A_1 \in \mathcal{N}$. Hence, $B \in \mathscr{B}^*$.

(e)

Show $\mathscr{B}^*$ is complete with respect to $\mathbf{P}^*$. Thus every σ-field has a completion.

Solution: Let $A \cup M \in \mathscr{B}^*$ where $A \in \mathscr{B}$ and $M \in \mathcal{N}$ be such that $\mathbf{P}^*(A \cup M) = 0$. We want to show every negligible set of $A \cup M$ is in $\mathscr{B}^*$, i.e. for any $E \subseteq A \cup M$, we have $E \in \mathscr{B}^*$, i.e. E is a $\mathscr{B}^*$-measurable set.

Firstly, see that

$$\mathbf{P}(A) = \mathbf{P}^*(A \cup M) = 0$$

by assumption.

Since $M \in \mathcal{N}$, there exists a $\mathbf{P}$-null set $N \in \mathscr{B}$ such that $M \subseteq N$. Hence, we have

$$E \subseteq A \cup M \subseteq A \cup N$$

where $A \cup N \in \mathscr{B}$. Because

$$\mathbf{P}(A \cup N) = \underbrace{\mathbf{P}(A)}_{=0} + \underbrace{\mathbf{P}(N)}_{=0} - \underbrace{\mathbf{P}(A \cap N)}_{\leqslant \mathbf{P}(N)=0} = 0$$

Hence, E is also a $\mathbf{P}$-negligible set.

In addition, see that $\varnothing \subseteq E$. And we know $\varnothing \in \mathscr{B}$ by definition of a σ-algebra. Thus, we have the following chain of inclusions

$$\varnothing \subseteq E \subseteq A \cup N$$

Also observe that

$$\mathbf{P}(A \cup N \backslash \varnothing) = \mathbf{P}(A \cup N) = 0$$

Thus, by part (d) of this question, we have $E \in \mathscr{B}^*$. This completes the proof of $\mathscr{B}^*$ being complete.

Problem 2.6.27 (Regular measures)

Consider the probability space $(\mathbb{R}^k, \mathscr{B}(\mathbb{R}^k), \mathbf{P})$. A Borel set A is regular if

$$\mathbf{P}(A) = \inf\{\mathbf{P}(G) : G \supseteq A, G \text{ open}\}$$

and

$$\mathbf{P}(A) = \sup\{\mathbf{P}(F) : F \subseteq A, F \text{ closed}\}$$

$\mathbf{P}$ is regular if all Borel sets are regular. Define $\mathcal{C}$ to be the collection of regular sets.

(a)

Show $\mathbb{R}^k \in \mathcal{C}, \varnothing \in \mathcal{C}$.
Solution:

Show $\mathbb{R}^k \in \mathcal{C}$. Let $A := \mathbb{R}^k$. We know $\mathbb{R}^k$ is both open and closed. See that $\mathbb{R}^k$ is the only open set that contains $A = \mathbb{R}^k$. Thus, we have

$$\mathbf{P}(A) = \mathbf{P}(\mathbb{R}^k) = \inf\{\mathbf{P}(G) : G \supseteq A, G \text{ open}\}$$

In addition, $\mathbb{R}^k$ is the largest closed set that is contained in A. Hence,

$$\mathbf{P}(A) = \mathbf{P}(\mathbb{R}^k) = \sup\{\mathbf{P}(F) : F \subseteq A, F \text{ closed}\}$$

Thus, we have $\mathbb{R}^k \in \mathcal{C}$.

Show $\varnothing \in \mathcal{C}$. Denote $A = \varnothing$. We know $\varnothing$ is both open and closed. Observe that $\varnothing$ is the only closed set that is contained in $A = \varnothing$. Hence, we have

$$\mathbf{P}(A) = \mathbf{P}(\varnothing) = \sup\{\mathbf{P}(F) : F \subseteq A, F \text{ closed}\}$$

In addition, $\varnothing$ is the smallest open set that contains $A = \varnothing$. Hence,

$$\mathbf{P}(A) = \mathbf{P}(\varnothing) = \inf\{\mathbf{P}(G) : G \supseteq A, G \text{ open}\}$$

(b)

Show $\mathcal{C}$ is closed under complements and countable unions.
Solution:

Show $\mathcal{C}$ is closed under complements. Let $A \in \mathcal{C}$. Then we have

$$\mathbf{P}(A) = \inf\{\mathbf{P}(G) : G \supseteq A, G \text{ open}\}$$

and

$$\mathbf{P}(A) = \sup\{\mathbf{P}(F) : F \subseteq A, F \text{ closed}\}$$

We want to show $A^c \in \mathcal{C}$, i.e.

1) $\mathbf{P}(A^c) = \inf\{\mathbf{P}(G) : G \supseteq A^c, G \text{ open}\}$
2) $\mathbf{P}(A^c) = \sup\{\mathbf{P}(F) : F \subseteq A^c, F \text{ closed}\}$

Show $\mathbf{P}(A^c) = \sup\{\mathbf{P}(F) : F \subseteq A^c, F$ *closed*$\}$. Let's first focus on

$$\mathbf{P}(A) = \inf\{\mathbf{P}(G) : G \supseteq A, G \text{ open}\}$$

This means $\mathbf{P}(A)$ is the greatest lower bound of $\mathbf{P}(G)$ for all open sets $G \subseteq \mathbb{R}^k$ that contains A. Then for every $\epsilon > 0$, we will have at least one $G \subseteq \mathbb{R}^k$ open such that

$$\mathbf{P}(G) < \mathbf{P}(A) + \epsilon$$

Consider all open set $U \subseteq \mathbb{R}^k$ that contains A, i.e. $U \supseteq A$. Taking the complement, denote $F = U^c$. Firstly, F are all closed sets. Secondly, we are going to have $F = U^c \subseteq A^c$. Hence, by monotonicity of probability measure, we have

$$\mathbf{P}(F) \leqslant \mathbf{P}(A^c)$$

i.e. $\mathbf{P}(A^c)$ is an upper bound of $\mathbf{P}(F)$ for all closed sets $F = U^c \subseteq \mathbb{R}^k$ that is contained in A^c. This means we obtained all closed sets contained in A^c. Thirdly, if we focus on the closed set G^c, from $\mathbf{P}(G) \leqslant \mathbf{P}(A) + \epsilon$, we will have

$$\mathbf{P}(G^c) > \mathbf{P}(A^c) - \epsilon$$

Thus, $\mathbf{P}(A^c)$ is the least upper bound of $\mathbf{P}(U^c)$ for all closed sets $F = U^c \subseteq \mathbb{R}^k$ that is contained in A^c. Hence, we have

$$\mathbf{P}(A^c) = \sup\{\mathbf{P}(F) : F \subseteq A^c, F \text{ closed}\}$$

Show $\mathbf{P}(A^c) = \inf\{\mathbf{P}(G) : G \supseteq A^c, G$ *open*$\}$. Let's first focus on

$$\mathbf{P}(A) = \sup\{\mathbf{P}(F) : F \subseteq A, F \text{ closed}\}$$

This means $\mathbf{P}(A)$ is the least upper bound of $\mathbf{P}(C)$ for all closed sets $C \subseteq \mathbb{R}^k$ that is contained in A. Then for every $\epsilon > 0$, we will have at least one $F \subseteq \mathbb{R}^k$ closed such that

$$\mathbf{P}(F) > \mathbf{P}(A) - \epsilon$$

Consider all closed set $C \subseteq \mathbb{R}^k$ that is contained in A, i.e. $C \subseteq A$. Taking the complement, denote $G = C^c$. Firstly, G are all open sets. Secondly, we are going to have $G = C^c \supseteq A^c$. Hence, by monotonicity of probability measure, we have

$$\mathbf{P}(G) \geqslant \mathbf{P}(A^c)$$

i.e. $\mathbf{P}(A^c)$ is a lower bound of $\mathbf{P}(G)$ for all open sets $G = C^c \in \mathbb{R}^k$ that contains A^c. This means we obtained all closed sets contained in A^c. Thirdly, if we focus on the closed set F^c, from $\mathbf{P}(F) > \mathbf{P}(A) - \epsilon$, we will have

$$\mathbf{P}(F^c) < \mathbf{P}(A^c) + \epsilon$$

Thus, $\mathbf{P}(A^c)$ is the greatest lower bound of $\mathbf{P}(C^c)$ for all open sets $G = C^c \in \mathbb{R}^k$ that contains A^c. Hence, we have

$$\mathbf{P}(A^c) = \inf\{\mathbf{P}(G) : G \supseteq A^c, G \text{ open}\}$$

Show $\mathcal{C}$ is closed under countable unions. Let $A_1, \ldots, A_n, \ldots \in \mathcal{C}$. Want to show $\cup_{n=1}^{\infty} A_n \in \mathcal{C}$, i.e.

1) $\mathbf{P}(\cup_{n=1}^{\infty} A_n) = \inf\{\mathbf{P}(G) : G \supseteq \cup_{n=1}^{\infty} A_n, G \text{ open}\}$

2) $\mathbf{P}(\cup_{n=1}^{\infty} A_n) = \sup\{\mathbf{P}(F) : F \subseteq \cup_{n=1}^{\infty} A_n, F \text{ closed}\}$

Denote $A = \cup_{n=1}^{\infty} A_n$.

Show $\mathbf{P}(\cup_{n=1}^{\infty} A_n) = \inf\{\mathbf{P}(G) : G \supseteq \cup_{n=1}^{\infty} A_n, G \textit{ open}\}$. We want to show $\mathbf{P}(\cup_{n=1}^{\infty} A_n)$ is the greatest lower bound of $\mathbf{P}(U)$ where $U \subseteq \mathbb{R}^k$ is an open set that contains A, i.e. $U \supseteq A$.

Let $\epsilon > 0$. For each $n \in \mathbb{Z}_{\geqslant 1}$, there exists an open set $U_n \subseteq \mathbb{R}^k$ such that $A_n \subseteq U_n$ and

$$\mathbf{P}(U_n) < \mathbf{P}(A_n) + \frac{\epsilon}{2^n}$$

Denote $U = \cup_{n=1}^{\infty} U_n$ which is an countable union of open sets, which is also open. Then we have $\cup_{n=1}^{\infty} A_n \subseteq \cup_{n=1}^{\infty} U_n$. Hence, we have

$$\mathbf{P}(A) = \mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \mathbf{P}\left(\bigcup_{n=1}^{\infty} U_n\right) = \mathbf{P}(U)$$

Observe that

$$\begin{aligned}
U \backslash A &= \left(\bigcup_{n=1}^{\infty} U_n\right) \backslash \left(\bigcup_{n=1}^{\infty} A_n\right) \\
&= \left(\bigcup_{n=1}^{\infty} U_n\right) \cap \left(\bigcap_{n=1}^{\infty} A_n^c\right), \\
&= \left(\bigcup_{n=1}^{\infty} \left(U_n \cap \underbrace{\bigcap_{k=1}^{\infty} A_k^c}_{\subset A_n^c}\right)\right), \qquad \text{by distributive law} \\
&\subseteq \bigcup_{n=1}^{\infty} (U_n \cap A_n^c)
\end{aligned}$$

Then by monotonicity and countable sub-additivity, we have

$$\begin{aligned}
\mathbf{P}(U) - \mathbf{P}(A) = \mathbf{P}(U\backslash A) &= \mathbf{P}\left(\left(\bigcup_{n=1}^{\infty} U_n\right)\backslash\left(\bigcup_{n=1}^{\infty} A_n\right)\right) \\
&\leqslant \mathbf{P}\left(\bigcup_{n=1}^{\infty}(U_n \cap A_n^c)\right) \\
&\leqslant \sum_{n=1}^{\infty} \mathbf{P}(U_n \cap A_n^c) \\
&= \sum_{n=1}^{\infty} \mathbf{P}(U_n\backslash A_n) \\
&= \sum_{n=1}^{\infty} (\mathbf{P}(U_n) - \mathbf{P}(A_n)) \\
&< \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \\
&= \epsilon
\end{aligned}$$

Hence, we get

$$\mathbf{P}(U) < \mathbf{P}(A) + \epsilon$$

This shows $\mathbf{P}(A)$ is indeed a greatest lower bound.

Show $\mathbf{P}(\cup_{n=1}^{\infty} A_n) = \sup\{\mathbf{P}(F) : F \subseteq \cup_{n=1}^{\infty} A_n, F \textit{ closed}\}$. Similarly, we want to show that $\mathbf{P}(A)$ is the least upper bound of $\mathbf{P}(C)$ where $C \subseteq \mathbb{R}^k$ is a closed set that is contained in A, i.e. $C \subseteq A$.

Let $\epsilon > 0$. For each $n \in \mathbb{Z}_{\geqslant 1}$, there exists a closed set $C_n \subseteq \mathbb{R}^k$ such that $A_n \supseteq C_n$ and

$$\mathbf{P}(C_n) > \mathbf{P}(A_n) - \frac{\epsilon}{2^n}$$

Denote $C = \cap_{n=1}^{\infty} C_n$ which is a countable intersection of closed sets, which is also closed. Then we have $\cup_{n=1}^{\infty} A_n \supseteq \cap_{n=1}^{\infty} C_n$. Hence, we have

$$\mathbf{P}(A) = \mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \geqslant \mathbf{P}\left(\bigcap_{n=1}^{\infty} C_n\right) = \mathbf{P}(C)$$

Observe that

$$\begin{aligned}
A\backslash C &= \left(\bigcup_{n=1}^{\infty} A_n\right)\backslash\left(\bigcap_{n=1}^{\infty} C_n\right) \\
&= \left(\bigcup_{n=1}^{\infty} A_n\right) \cap C^c \\
&= \bigcup_{n=1}^{\infty}(A_n \cap \underbrace{C^c}_{\subseteq C_n^c}), \qquad \text{by distributive law} \\
&\subseteq \bigcup_{n=1}^{\infty}(A_n \cap C_n^c)
\end{aligned}$$

Then by monotonicity and countable sub-additivity, we have

$$\begin{aligned}
\mathbf{P}(A) - \mathbf{P}(C) = \mathbf{P}(A \backslash C) &= \mathbf{P}\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \backslash \left(\bigcap_{n=1}^{\infty} C_n\right)\right) \\
&\leqslant \mathbf{P}\left(\bigcup_{n=1}^{\infty} (A_n \cap C_n^c)\right) \\
&\leqslant \sum_{n=1}^{\infty} \mathbf{P}(A_n \cap C_n^c) \\
&= \sum_{n=1}^{\infty} \mathbf{P}(A_n \backslash C_n) \\
&= \sum_{n=1}^{\infty} (\mathbf{P}(A_n) - \mathbf{P}(C_n)) \\
&< \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \\
&= \epsilon
\end{aligned}$$

Hence, we get

$$\mathbf{P}(C) \geqslant \mathbf{P}(A) - \epsilon$$

This shows $\mathbf{P}(A)$ is indeed a least upper bound.

(c)

Let $\mathcal{F}(\mathbb{R}^k)$ be the collection of closed subsets of $\mathbb{R}^k$. Show

$$\mathcal{F}(\mathbb{R}^k) \subset \mathcal{C}.$$

Solution: Let $C \in \mathcal{F}(\mathbb{R}^k)$. Because C is closed, it is clear that $C \subseteq C$ is closed. Hence,

$$\mathbf{P}(C) = \sup\{\mathbf{P}(F) : F \subseteq C, F \text{ closed}\}$$

Now I want to show

$$\mathbf{P}(C) = \inf\{\mathbf{P}(G) : G \supseteq C, G \text{ open}\}$$

In particular, we want to show for all $\epsilon > 0$, there is an open set $U \subseteq \mathbb{R}^k$ where $U \supseteq C$ such that

$$\mathbf{P}(U) < \mathbf{P}(C) + \epsilon$$

Show $\mathbf{P}(K) = \inf\{\mathbf{P}(G) : G \supseteq K, G$ open$\}$ for $K \subseteq \mathbb{R}^k$ compact. First, let's show this statement holds for a compact set $K \subset \mathbb{R}^k$. For $m \in \mathbb{Z}_{\geqslant 1}$ let

$$U_m := \left\{x \in \mathbb{R}^k : d(x, K) < \tfrac{1}{m}\right\}$$

be the open $\frac{1}{m}$-neighborhood of K. Each U_m is open and contains K, the sequence $(U_m)_{m \geqslant 1}$ is decreasing, and $\bigcap_{m \geqslant 1} U_m = K$, because a point at distance 0 from the closed set K belongs to K. By continuity from above of $\mathbf{P}$,

$$\mathbf{P}(U_m) \longrightarrow \mathbf{P}(K) \qquad \text{as } m \to \infty$$

so for every $\epsilon > 0$ there is an open set $U := U_m \supseteq K$ (for m large enough) such that

$$\mathbf{P}(U \backslash K) = \mathbf{P}(U) - \mathbf{P}(K) < \epsilon$$

Hence, we have

$$\mathbf{P}(U) \leqslant \mathbf{P}(K) + \epsilon$$

Hence, $\mathbf{P}(K)$ is a greatest lower bound of $\mathbf{P}(G)$ for $G \in \mathbb{R}^k$ open that contains K. Hence, we have

$$\mathbf{P}(K) = \inf\{\mathbf{P}(G) : G \supseteq K, G \text{ open}\}$$

Hence, we will have $K \in \mathcal{C}$ for all $K \subset \mathbb{R}^k$ compact.

Show $\mathbf{P}(C) = \inf\{\mathbf{P}(G) : G \supseteq C, G$ open$\}$ for $C \subseteq \mathbb{R}^k$ closed. For $C \subseteq \mathbb{R}^k$ closed, consider the intersection with all closed balls. See that

$$C = \bigcup_{r=1}^{\infty} (C \cap \overline{B_r(0)})$$

where $B_r(0) = \{x \in \mathbb{R}^k : d(x, 0) < r\}$ is the ball centered at origin of radius r, and $\overline{B_r(0)}$ means the closure of this open ball, which will be compact. Then $C \cap \overline{B_r(0)}$ will be compact. Because we already have showed $K \in \mathcal{C}$ for all $K \subset \mathbb{R}^k$ compact, and we showed $\mathcal{C}$ is a σ-algebra. Thus, we know countable union of compact sets will also be in $\mathcal{C}$. Hence,

$$C = \bigcup_{r=1}^{\infty} (C \cap \overline{B_r(0)}) \in \mathcal{C}$$

(d)

Show $\mathscr{B}(\mathbb{R}^k) \subset C$; that is, show regularity.

Solution: Because Borel σ-algebra $\mathscr{B}(\mathbb{R}^k)$ is generated by all open sets of $\mathbb{R}^k$, i.e.

$$\mathscr{B}(\mathbb{R}^k) = \sigma(\{U \subset \mathbb{R}^k : U \text{ open}\})$$

Similarly, we can say Borel σ-algebra $\mathscr{B}(\mathbb{R}^k)$ is generated by all closed sets of $\mathbb{R}^k$, i.e.

$$\mathscr{B}(\mathbb{R}^k) = \sigma(\{C \subset \mathbb{R}^k : C \text{ closed}\}) = \sigma(\mathcal{F}(\mathbb{R}^k))$$

Hence, Borel σ-algebra $\mathscr{B}(\mathbb{R}^k)$ is the smallest σ-algebra that contains $\mathcal{F}(\mathbb{R}^k)$. Because the set of regular set of $\mathbb{R}^k$ is also a σ-algebra, which we denote as $\mathcal{C}$. Hence we have

$$\mathscr{B}(\mathbb{R}^k) \subset \mathcal{C}$$

(e)

For any Borel set A

$$\mathbf{P}(A) = \sup\{\mathbf{P}(K) : K \subset A, K \text{ compact}\}$$

Solution: Pick a closed set. Precisely, let $A \in \mathscr{B}(\mathbb{R}^k)$ and let $\epsilon > 0$. By Part (d) the set A is regular, so in particular

$$\mathbf{P}(A) = \sup\{\mathbf{P}(F) : F \subseteq A, F \text{ closed}\}$$

and hence there is a closed set $F \subseteq A$ with

$$\mathbf{P}(F) > \mathbf{P}(A) - \frac{\epsilon}{2}$$

Now exhaust F by compact sets. For $n \in \mathbb{Z}_{\geqslant 1}$ put

$$K_n := F \cap \overline{B_n(0)}, \qquad \overline{B_n(0)} := \{x \in \mathbb{R}^k : d(x, 0) \leqslant n\}$$

Each K_n is an intersection of two closed sets, hence closed, and it is bounded; so K_n is compact by Heine–Borel, and $K_n \subseteq F \subseteq A$. The sequence (K_n) is increasing with

$$\bigcup_{n \geqslant 1} K_n = F \cap \bigcup_{n \geqslant 1} \overline{B_n(0)} = F \cap \mathbb{R}^k = F$$

so continuity from below of $\mathbf{P}$ gives

$$\mathbf{P}(K_n) \longrightarrow \mathbf{P}(F) \qquad \text{as } n \to \infty$$

Choose n with $\mathbf{P}(K_n) > \mathbf{P}(F) - \frac{\epsilon}{2}$. Combining the two estimates,

$$\mathbf{P}(K_n) > \mathbf{P}(F) - \frac{\epsilon}{2} > \mathbf{P}(A) - \epsilon, \qquad K_n \subseteq A, \quad K_n \text{ compact}$$

Hence

$$\sup\{\mathbf{P}(K) : K \subseteq A, K \text{ compact}\} \geqslant \mathbf{P}(A) - \epsilon$$

and since $\epsilon > 0$ was arbitrary, the supremum is $\geqslant \mathbf{P}(A)$. The reverse inequality is immediate from monotonicity: $K \subseteq A$ gives $\mathbf{P}(K) \leqslant \mathbf{P}(A)$, so $\mathbf{P}(A)$ is an upper bound for the set of numbers $\mathbf{P}(K)$. Therefore

$$\mathbf{P}(A) = \sup\{\mathbf{P}(K) : K \subseteq A, K \text{ compact}\}$$

for every Borel set A, which is the assertion.

Problem 3.4.2

Let $(\Omega, \mathscr{B}, \mathbf{P}) = ((0, 1], \mathscr{B}((0, 1]), \lambda)$ where λ is Lebesgue measure. Define

$$\begin{aligned} X_1(\omega) &= 0, \qquad \forall \omega \in \Omega \\ X_2(\omega) &= 1_{\{1/2\}}(\omega), \\ X_3(\omega) &= 1_{\mathbb{Q} \cap (0,1])}(\omega) \end{aligned}$$

where $\mathbb{Q} \cap (0, 1] \subset (0, 1]$ are the rational numbers in $(0, 1]$. Note

$$\mathbf{P}(X_1 = X_2 = X_3 = 0) = 1$$

and give

$$\sigma(X_i), \qquad i = 1, 2, 3.$$

Solution:

Part (1). Show $\mathbf{P}(X_1 = X_2 = X_3 = 0) = 1$. Note that the event in question is an intersection,

$$\{X_1 = X_2 = X_3 = 0\} = \{X_1 = 0\} \cap \{X_2 = 0\} \cap \{X_3 = 0\}$$

and each of the three events is immediate from the definitions:

$$\begin{aligned}
\{X_1 = 0\} &= \Omega = (0,1] \\
\{X_2 = 0\} &= \big\{\omega \in (0,1] : 1_{\{1/2\}}(\omega) = 0\big\} = (0,1]\backslash\{1/2\} \\
\{X_3 = 0\} &= \big\{\omega \in (0,1] : 1_{\mathbb{Q}\cap(0,1]}(\omega) = 0\big\} = (0,1]\backslash\mathbb{Q}
\end{aligned}$$

where in the last line $\mathbb{Q}$ abbreviates $\mathbb{Q} \cap (0,1]$. Since $\frac{1}{2} \in \mathbb{Q} \cap (0,1]$, we have $(0,1]\backslash\mathbb{Q} \subseteq (0,1]\backslash\{1/2\}$, so the intersection collapses to

$$\{X_1 = X_2 = X_3 = 0\} = (0,1]\backslash\mathbb{Q}$$

which is a Borel subset of $(0,1]$, being the complement of a countable set.

It remains to compute its Lebesgue measure. The set $\mathbb{Q} \cap (0,1]$ is countable, say $\mathbb{Q} \cap (0,1] = \{q_i : i \geqslant 1\}$, and each singleton is Lebesgue null, $\lambda(\{q_i\}) = 0$. By countable sub-additivity,

$$\lambda(\mathbb{Q} \cap (0,1]) \leqslant \sum_{i\geqslant 1} \lambda(\{q_i\}) = 0$$

Hence

$$\mathbf{P}(X_1 = X_2 = X_3 = 0) = \lambda((0,1]\backslash\mathbb{Q}) = \lambda((0,1]) - \lambda(\mathbb{Q} \cap (0,1]) = 1 - 0 = 1$$

Part (2). First see that each X_i is a function

$$X_i : (\Omega, \mathscr{B}, \mathbf{P}) \longrightarrow (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbf{P}' = \mathbf{P} \circ X_i^{-1})$$

Find $\sigma(X_1)$. See that for any $B \in \mathscr{B}(\mathbb{R})$, we have

$$X_1^{-1}(B) = \begin{cases} \Omega, & 0 \in B \\ \varnothing, & 0 \notin B \end{cases}$$

Hence, $\sigma(X_1) := X_1^{-1}(\mathscr{B}(\mathbb{R})) = \{\Omega, \varnothing\}$.

Find $\sigma(X_2)$. See that for any $B \in \mathscr{B}(\mathbb{R})$, we have

$$X_2^{-1}(B) = \begin{cases} \Omega, & 0 \in B,\ 1 \in B \\ \varnothing, & 0 \notin B,\ 1 \notin B \\ \{1/2\}, & 0 \notin B,\ 1 \in B \\ (0,1]\backslash\{1/2\}, & 0 \in B,\ 1 \notin B \end{cases}$$

Hence, $\sigma(X_2) := X_2^{-1}(\mathscr{B}(\mathbb{R})) = \{\Omega, \varnothing, \{1/2\}, (0,1]\backslash\{1/2\}\}$.

Find $\sigma(X_3)$. See that for any $B \in \mathscr{B}(\mathbb{R})$, we have

$$X_3^{-1}(B) = \begin{cases} \Omega, & 0 \in B,\ 1 \in B \\ \varnothing, & 0 \notin B,\ 1 \notin B \\ \mathbb{Q} \cap (0,1], & 0 \notin B,\ 1 \in B \\ (0,1]\backslash\mathbb{Q}, & 0 \in B,\ 1 \notin B \end{cases}$$

Hence, $\sigma(X_3) := X_3^{-1}(\mathscr{B}(\mathbb{R})) = \{\Omega, \varnothing, \mathbb{Q} \cap (0,1], (0,1]\backslash\mathbb{Q}\}$.

8 Homework 8

HW8 (due on 11/06): 3.4 - 4, 5, 6, 17.

Problem 3.4.4

Suppose $X : \Omega \to \mathbb{R}$ has a countable range $\mathcal{R}$. Show that $X \in \mathscr{B}/\mathscr{B}(\mathbb{R})$ if and only if

$$X^{-1}(\{x\}) \in \mathscr{B}, \qquad \forall x \in \mathcal{R}$$

Solution:

$\Longrightarrow$

Suppose $X \in \mathscr{B}/\mathscr{B}(\mathbb{R})$. Then the claim is $\mathcal{R} \subset \mathscr{B}(\mathbb{R})$. It is clear that for any $x \in \mathcal{R}$, $\{x\} \in \mathscr{B}(\mathbb{R})$ because $\{x\} \subset \mathbb{R}$ is a closed set. Then by $X \in \mathscr{B}/\mathscr{B}(\mathbb{R})$, we have $X^{-1}(\{x\}) \in \mathscr{B}$.

$\Longleftarrow$

Suppose for all $x \in \mathcal{R}$, $X^{-1}(\{x\}) \in \mathscr{B}$. Want to show $X \in \mathscr{B}/\mathscr{B}(\mathbb{R})$, i.e. $X^{-1}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}$. Let $A \in \mathscr{B}(\mathbb{R})$. Because of $\mathscr{B}$ is a σ-algebra (which means it is closed under countable union), then we have

$$X^{-1}(A) = X^{-1}(\mathcal{R} \cap A) = \bigcup_{x \in A \cap \mathcal{R}} \underbrace{X^{-1}(\{x\})}_{\in \mathscr{B}} \in \mathscr{B}$$

Thus, we conclude $X \in \mathscr{B}/\mathscr{B}(\mathbb{R})$.

Problem 3.4.5

If $F(x) = \mathbf{P}(X \leqslant x)$ is continuous in x. Show that $Y = F(X)$ is measurable and that Y has a uniform distribution

$$\mathbf{P}(Y \leqslant y) = y, \qquad 0 \leqslant y \leqslant 1$$

Solution:

Measurability of Y. Consider the map between measurable spaces

$$\begin{array}{rcccccc} Y = F(X): & (\Omega, \mathscr{B}) & \longrightarrow & (\mathbb{R}, \mathscr{B}(\mathbb{R})) & \longrightarrow & ([0,1], \mathscr{B}([0,1])) \\ & \omega & \longmapsto & X(\omega) & \longmapsto & F(X(\omega)) \end{array}$$

Want to show $Y^{-1}(\mathscr{B}([0,1])) \subset \mathscr{B}$. We already showed composition of measurable functions is measurable. So we only need to show $F^{-1}(\mathscr{B}([0,1])) \subset \mathscr{B}(\mathbb{R})$.

Denote $\mathcal{T}([0,1])$ be the set of open sets of $[0,1]$, and $\mathcal{T}(\mathbb{R})$ be the set of open sets of $\mathbb{R}$. We know $\sigma(\mathcal{T}([0,1])) = \mathscr{B}([0,1])$, and $\sigma(\mathcal{T}(\mathbb{R})) = \mathscr{B}(\mathbb{R})$. Because if we want to show

$$\sigma(F^{-1}(\mathcal{T}([0,1]))) = F^{-1}(\sigma(\mathcal{T}([0,1]))) = F^{-1}(\mathscr{B}([0,1])) \subset \mathscr{B}(\mathbb{R})$$

using the definition of the smallest σ-algebra, it suffices to show $F^{-1}(\mathcal{T}([0,1])) \subset \mathscr{B}(\mathbb{R})$. Let $A \in \mathcal{T}([0,1])$. Because F is continuous at x, thus

$$F^{-1}(A) \in \mathcal{T}(\mathbb{R}) \subset \sigma(\mathcal{T}(\mathbb{R})) = \mathscr{B}(\mathbb{R})$$

Hence, we conclude $F^{-1}(\mathcal{T}([0,1])) \subset \mathscr{B}(\mathbb{R})$. Thus, $F^{-1}(\mathscr{B}([0,1])) \subset \mathscr{B}(\mathbb{R})$. This completes the proof.

Distribution of Y. We need to show

$$\mathbf{P}(Y \leqslant y) = y, \qquad \forall y \in [0,1]$$

Fix $y \in (0,1)$ and set

$$x_y := \sup\{t \in \mathbb{R} : F(t) \leqslant y\}$$

which is finite because $F(t) \to 0$ as $t \to -\infty$ and $F(t) \to 1$ as $t \to +\infty$. By continuity of F we have $F(x_y) = y$: indeed $F(t) \leqslant y$ for all $t < x_y$ gives $F(x_y) \leqslant y$ by continuity, while $F(t) > y$ for all $t > x_y$ gives $F(x_y) \geqslant y$. Moreover

$$\{Y \leqslant y\} = \{F(X) \leqslant y\} = \{X \leqslant x_y\}$$

because $X \leqslant x_y$ implies $F(X) \leqslant F(x_y) = y$ by monotonicity, while $X > x_y$ implies $F(X) > y$ by the definition of the supremum. Then

$$\mathbf{P}(Y \leqslant y) = \mathbf{P}(X \leqslant x_y) = F(x_y) = y$$

For the endpoints: $\mathbf{P}(Y \leqslant 1) = 1$ trivially. For $y = 0$, note $\{Y \leqslant 0\} = \{F(X) = 0\} = \{X \in S_0\}$ where $S_0 := \{t : F(t) = 0\}$ is closed by continuity; if $S_0 = \varnothing$ the event is empty, and otherwise S_0 is bounded above (since $F \to 1$) with maximum $t_0 \in S_0$, so

$$\mathbf{P}(Y \leqslant 0) = \mathbf{P}(X \in S_0) \leqslant \mathbf{P}(X \leqslant t_0) = F(t_0) = 0$$

Either way $\mathbf{P}(Y \leqslant 0) = 0$, completing the verification.

Problem 3.4.6

If X is a random variable satisfying $\mathbf{P}(|X| < \infty) = 1$, then show that for any $\epsilon > 0$, there exists a bounded random variable Y such that

$$\mathbf{P}(X \neq Y) < \epsilon$$

(A random variable Y is bounded if for all ω

$$|Y(\omega)| \leqslant K$$

for some constant K independent of ω.)

Solution: Let $\epsilon > 0$. Because $\mathbf{P}(|X| < \infty) = 1$. There exists a large constant $K \geqslant 0$ such that

$$\mathbf{P}(|X| > K) < \epsilon$$

Define Y as the truncation of X at K.

$$Y = X\mathbf{1}_{|X| \leqslant K}$$

Clearly Y is bounded. See that

$$\{X \neq Y\} = \{|X| > K\}$$

Then

$$\mathbf{P}(X \neq Y) = \mathbf{P}(|X| > K) < \epsilon$$

Problem 3.4.17

Functions are often defined in pieces (for example, let $f(x)$ be x^3 or x^{-1} as $x \geqslant 0$ or $x < 0$), and the following shows that the function is measurable if the pieces are.

Consider measurable spaces $(\Omega, \mathscr{B})$ and $(\Omega', \mathscr{B}')$ and a map $T : \Omega \to \Omega'$. Let $A_1, A_2, \ldots$ be a countable covering of Ω by $\mathscr{B}$ sets. Consider the σ-field $\mathscr{B}_n = \{A : A \subset A_n, A \in \mathscr{B}\}$ in A_n and the restriction T_n of T to A_n.

Show that T is measurable $\mathscr{B}/\mathscr{B}'$ iff T_n is measurable $\mathscr{B}_n/\mathscr{B}'$ for each n.

Solution:

$\Longrightarrow$

Suppose $T \in \mathscr{B}/\mathscr{B}'$. Choose an arbitrary n. Let $B \in \mathscr{B}'$. We want to show $T_n^{-1}(B) \in \mathscr{B}_n$. See that

$$T_n^{-1}(B) = \underbrace{\underbrace{T^{-1}(B)}_{\in \mathscr{B}} \cap \underbrace{A_n}_{\in \mathscr{B}}}_{\subset A_n} \in \mathscr{B}_n$$

This completes the proof.

$\Longleftarrow$

Suppose $T_n \in \mathscr{B}_n/\mathscr{B}'$ for all n. We want to show $T^{-1}(\mathscr{B}') \subset \mathscr{B}$. Let $B \in \mathscr{B}'$. Because $(A_n)_{n \geqslant 1}$ is a $\mathscr{B}$-cover of Ω, observe that

$$T^{-1}(B) = T^{-1}(B) \cap \left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \underbrace{(T^{-1}(B) \cap A_n)}_{=T_n^{-1}(B)} = \bigcup_{n=1}^{\infty} \underbrace{T_n^{-1}(B)}_{\in \mathscr{B}_n \subset \mathscr{B}} \in \mathscr{B}$$

This completes the proof.

9 Homework 9

HW9 (due on 11/13): 3.4 - 18, 22; 4.6 - 1, 3.

Question 3.4.18 (Coupling)

If X and Y are random variables on $(\Omega, \mathscr{B})$, show

$$\sup_{A \in \mathscr{B}} |\mathbf{P}(X \in A) - \mathbf{P}(Y \in A)| \leqslant \mathbf{P}(X \neq Y)$$

Solution: Observe that for any $A \in \mathscr{B}$,

$$\begin{aligned}
\mathbf{P}(X \in A) - \mathbf{P}(Y \in A) =\ & \mathbf{P}(X \in A, X = Y) + \mathbf{P}(X \in A, X \neq Y) \\
& - \mathbf{P}(Y \in A, X = Y) - \mathbf{P}(Y \in A, X \neq Y) \\
=\ & \underbrace{\mathbf{P}(X \in A, X = Y) - \mathbf{P}(Y \in A, X = Y)}_{=0} \\
& + \mathbf{P}(X \in A, X \neq Y) - \mathbf{P}(Y \in A, X \neq Y) \\
=\ & \mathbf{P}(X \in A, X \neq Y) - \mathbf{P}(Y \in A, X \neq Y)
\end{aligned}$$

Hence, for any $A \in \mathscr{B}$, we have

$$\begin{aligned}
|\mathbf{P}(X \in A) - \mathbf{P}(Y \in A)| &= |\mathbf{P}(X \in A, X \neq Y) - \mathbf{P}(Y \in A, X \neq Y)| \\
&\leqslant \max\{\mathbf{P}(X \in A, X \neq Y),\ \mathbf{P}(Y \in A, X \neq Y)\} \leqslant \mathbf{P}(X \neq Y)
\end{aligned}$$

because for non-negative a, b we have $|a-b| \leqslant \max\{a, b\}$, and both events $\{X \in A, X \neq Y\}$ and $\{Y \in A, X \neq Y\}$ are contained in $\{X \neq Y\}$, so both probabilities are at most $\mathbf{P}(X \neq Y)$.

Since this bound holds for every A and its right-hand side does not depend on A, taking the supremum over A gives

$$\sup_{A} |\mathbf{P}(X \in A) - \mathbf{P}(Y \in A)| \leqslant \mathbf{P}(X \neq Y)$$

which is the claim.

Question 3.4.22 (First up-crossing time)

Suppose $\{X_n : n \geqslant 1\}$ are random variables on the probability space $(\Omega, \mathscr{B}, \mathbf{P})$ and define the induced random walk by

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i, n \geqslant 1$$

Let

$$\tau := \inf\{n > 0 : S_n > 0\}$$

be the first upgoing ladder time. Prove τ is a random variable. Assume we know $\tau(\omega) < \infty$ for all $\omega \in \Omega$. Prove S_τ is a random variable.

Solution:

Show τ is a random variable. Want to show $\tau^{-1}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}$. Consider the event $(-\infty, n] \in \mathscr{B}(\mathbb{R})$ where $n > 0$ is arbitrary. Consider

$$\begin{aligned}
\tau^{-1}((-\infty, n]) = \{\omega \in \Omega : \tau(\omega) \in (-\infty, n]\} &= \{\omega \in \Omega : \tau(\omega) \leqslant n\} \\
&= \bigcup_{k=1}^{n} \{\omega \in \Omega : \tau(\omega) = k\} \\
&= \bigcup_{k=1}^{n} \Big(\{S_m \leqslant 0 \text{ for all } m < k\} \cap \{S_k > 0\} \Big) \\
&= \bigcup_{k=1}^{n} \Big(\bigcap_{m=1}^{k-1} \{S_m \leqslant 0\} \cap \{S_k > 0\} \Big) \\
&= \bigcup_{k=1}^{n} \Big(\bigcap_{m=1}^{k-1} \Big\{ \sum_{i=1}^{m} X_i \leqslant 0 \Big\} \cap \Big\{ \sum_{i=1}^{k} X_i > 0 \Big\} \Big)
\end{aligned}$$

Now the claim is

$$\Big\{ \sum_{i=1}^{m} X_i \leqslant 0 \Big\} \in \mathscr{B}, \qquad \Big\{ \sum_{i=1}^{k} X_i > 0 \Big\} \in \mathscr{B}$$

Because finite sum of random variable is still a random variable. (This is obvious for simple random variables. And the general random variable is the closure of set of simple random variables $\mathcal{E}$). Hence, we conclude

$$\tau^{-1}((-\infty, n]) = \begin{cases} \bigcup_{k=1}^{n} \Big(\bigcap_{m=1}^{k-1} \{\sum_{i=1}^{m} X_i \leqslant 0\} \cap \Big\{ \sum_{i=1}^{k} X_i > 0 \Big\} \Big) \in \mathscr{B}, & n > 0 \\ \varnothing, & n \leqslant 0 \end{cases}$$

This determines $\tau^{-1}((-\infty, x])$ for every real x as well: since τ takes values in $\{1, 2, \ldots\} \cup \{+\infty\}$, we have $\{\tau \leqslant x\} = \{\tau \leqslant \lfloor x \rfloor\}$, so $\tau^{-1}((-\infty, x]) \in \mathscr{B}$ for all $x \in \mathbb{R}$. The collection $\{(-\infty, x] : x \in \mathbb{R}\}$ generates $\mathscr{B}(\mathbb{R})$, so by the test for measurability we conclude τ is measurable.

Show S_τ is a random variable. Observe that

$$S_\tau = \sum_{n=1}^{\infty} S_n \mathbf{1}_{\{\tau = n\}}$$

Here S_n is a finite sum of random variables, hence is also a random variable.

- On the other hand, $\{\tau = n\} = \{\tau \leqslant n\} \backslash \{\tau \leqslant n - 1\} \in \mathscr{B}$. Hence, $\mathbf{1}_{\{\tau = n\}}$ is a measurable function.

- Product of measurable functions is a measurable function. This is obviously true for simple measurable functions. So we can find an approximating sequence of the product. Under this construction, product will be a measurable function. Hence, $S_n \cdot \mathbf{1}_{\{\tau = n\}}$ is a measurable function.

- Countable sum of measurable functions is a measurable. Because the partial sum
$$\sum_{n=1}^{N} S_n \mathbf{1}_{\{\tau=n\}}$$
will be measurable. We proved limit of measurable functions (if exists) is measurable. Hence, $S_\tau = \sum_{n=1}^{\infty} S_n \mathbf{1}_{\{\tau=n\}}$ is measurable. This finishes the proof.

Question 4.6.1

Let $B_1, \ldots, B_n$ be independent events. Show
$$\mathbf{P}\left(\bigcup_{i=1}^{n} B_i\right) = 1 - \prod_{i=1}^{n}(1 - \mathbf{P}(B_i))$$

Solution: See that by independence of events, we have
$$\begin{aligned}
\mathbf{P}\left(\bigcup_{i=1}^{n} B_i\right) &= 1 - \mathbf{P}\left(\left(\bigcup_{i=1}^{n} B_i\right)^c\right) \\
&= 1 - \mathbf{P}\left(\bigcap_{i=1}^{n} B_i^c\right) \\
&= 1 - \prod_{i=1}^{n} \mathbf{P}(B_i^c), \qquad \text{independence} \\
&= 1 - \prod_{i=1}^{n}(1 - \mathbf{P}(B_i))
\end{aligned}$$

Question 4.6.3

If $\{A_n : n \geqslant 1\}$ is an independent sequence of events, show
$$\mathbf{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathbf{P}(A_n)$$

Solution: Denote
$$B_n = \bigcap_{k=1}^{n} A_k$$
Then see that
$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n =: B$$
and $\{B_n : n \geqslant 1\}$ is a decreasing sequence of events.

Then by continuity of probability along the decreasing sequence (B_N), and by independence of the finite collections $A_1, \ldots, A_N$, we have

$$\begin{aligned}
\mathbf{P}\left(\bigcap_{n=1}^{\infty} A_n\right) &= \mathbf{P}\left(\bigcap_{N=1}^{\infty} B_N\right) = \lim_{N\to\infty} \mathbf{P}(B_N) \\
&= \lim_{N\to\infty} \mathbf{P}\left(\bigcap_{n=1}^{N} A_n\right) = \lim_{N\to\infty} \prod_{n=1}^{N} \mathbf{P}(A_n) \\
&= \prod_{n=1}^{\infty} \mathbf{P}(A_n)
\end{aligned}$$

10 Homework 10

HW10 (due on 11/22 Friday): 4.6 - 8, 13, 17(a).

Exercise 4.6.8

If X and Y are independent random variables and f, g are measurable and real valued, why are $f(X)$ and $g(Y)$ independent? (No calculation is necessary.)
Solution: Here, the compositions of the maps are

$$(\Omega, \mathscr{B}, \mathbf{P}) \xrightarrow{X,Y} (\mathbb{R}, \mathscr{B}(\mathbb{R})) \xrightarrow{f,g} (\mathbb{R}, \mathscr{B}(\mathbb{R}))$$

Suppose X, Y are independent. That means $\sigma(X) = X^{-1}(\mathscr{B}(\mathbb{R}))$ and $\sigma(Y) = Y^{-1}(\mathscr{B}(\mathbb{R}))$ are independent. This means for all events $A \in \sigma(X)$ and $B \in \sigma(Y)$, A and B are independent.

Because $f, g \in \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$ are measurable. Hence,

$$f^{-1}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}(\mathbb{R}), \qquad g^{-1}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}(\mathbb{R})$$

Thus,

$$\begin{aligned}\sigma(f \circ X) &= (f \circ X)^{-1}(\mathscr{B}(\mathbb{R})) = X^{-1}(f^{-1}(\mathscr{B}(\mathbb{R}))) \subset \sigma(X),\\ \sigma(g \circ Y) &= (g \circ Y)^{-1}(\mathscr{B}(\mathbb{R})) = Y^{-1}(g^{-1}(\mathscr{B}(\mathbb{R}))) \subset \sigma(Y)\end{aligned}$$

Thus, any $A \in \sigma(f \circ X)$ and any $B \in \sigma(g \circ Y)$ are independent. Thus, $\sigma(f \circ X)$ and $\sigma(g \circ Y)$ are independent, which means $f(X)$ and $g(Y)$ are independent.

Exercise 4.6.13

Let $\{X_n : n \geqslant 1\}$ be iid with $\mathbf{P}(X_1 = 1) = p = 1 - \mathbf{P}(X_1 = 0)$. What is the probability that the pattern $1, 0, 1$ appears infinitely often?

Hint: Let

$$A_k = \{X_k = 1, X_{k+1} = 0, X_{k+2} = 1\}$$

and consider $A_1, A_4, A_7, \ldots$
Solution: We want to show the event A_k will happen infinitely often. Because X_n's are IID, then

$$\begin{aligned}\mathbf{P}(A_k) &= \mathbf{P}(X_k = 1, X_{k+1} = 0, X_{k+2} = 1)\\ &= \mathbf{P}(X_k = 1)\,\mathbf{P}(X_{k+1} = 0)\,\mathbf{P}(X_{k+2} = 1)\\ &= p^2(1-p)\end{aligned}$$

Still, because X_n's are independent, then events A_{1+3k} for $k \geqslant 1$ are independent. Consider

$$\sum_{k=1}^{\infty} \mathbf{P}(A_{1+3k}) = \sum_{k=1}^{\infty} p^2(1-p)$$

Because $\lim_{n\to\infty} p^2(1-p) = p^2(1-p)$, so by the divergence test, this series diverges for $p \in (0,1)$. Thus, by Borel-zero-one law, we have

$$\mathbf{P}(A_k \text{ i.o.}) = 1$$

It remains to consider the endpoint cases. If $p = 0$ or $p = 1$, then $\mathbf{P}(A_k) = p^2(1-p) = 0$ for every k, so $\sum_k \mathbf{P}(A_k) < \infty$ and the Borel–Cantelli lemma gives $\mathbf{P}(A_k \text{ i.o.}) = 0$; indeed the pattern $1, 0, 1$ requires both a 1 and a 0, which is impossible when the X_n are almost surely constant. Hence

$$\mathbf{P}(A_k \text{ i.o.}) = \begin{cases} 1, & p \in (0,1) \\ 0, & p \in \{0,1\} \end{cases}$$

Exercise 4.6.17

Review Example 4.5.2

Exercise 4.6.17 (a)

Suppose $\{X_n : n \geqslant 1\}$ are iid random variables and suppose $\{a_n\}$ is a sequence of constants. Show

$$\mathbf{P}(\{X_n > a_n\} \text{ i.o.}) = \begin{cases} 0 & \text{iff } \sum_n \mathbf{P}(X_1 > a_n) < \infty, \\ 1 & \text{iff } \sum_n \mathbf{P}(X_1 > a_n) = \infty, \end{cases}$$

Solution: Because X_n's are IID RVs, thus the event $A_n := \{X_n > a_n\}$ are independent. By Borel-zero-one law, we know

$$\mathbf{P}(A_n \text{ i.o.}) = \begin{cases} 0, & \text{if } \sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty \\ 1, & \text{if } \sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty \end{cases}$$

Consider the part $\sum_n \mathbf{P}(X_1 > a_n) < \infty$. Suppose $\sum_n \mathbf{P}(X_1 > a_n) < \infty$ is true. Then $\lim_{n\to\infty} \mathbf{P}(X_1 > a_n) = 0$. Because X_n's are IID, hence $\mathbf{P}(X_1 > a_n) = \mathbf{P}(X_n > a_n)$. Thus, we will have

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \sum_{n=1}^{\infty} \mathbf{P}(X_n > a_n) = \sum_{n=1}^{\infty} \mathbf{P}(X_1 > a_n) < \infty$$

Then by Borel zero-one law, we have the statement.

On the other hand, use contradiction. Suppose $\mathbf{P}(A_n \text{ i.o.}) = 0$, but we have $\sum_n \mathbf{P}(X_1 > a_n) = \infty$. By IID, we have

$$\infty = \sum_{n=1}^{\infty} \underbrace{\mathbf{P}(X_1 > a_n)}_{=\mathbf{P}(X_n > a_n)} = \sum_{n=1}^{\infty} \mathbf{P}(X_n > a_n)$$

Then by Borel zero-one law, we have $\mathbf{P}(\{X_n > a_n\} \text{ i.o.}) = 1$, which contradicts the statement $\mathbf{P}(A_n \text{ i.o.}) = 0$.

Consider the part $\sum_n \mathbf{P}(X_1 > a_n) = \infty$. Suppose $\sum_n \mathbf{P}(X_1 > a_n) = \infty$. Then by IID, we have

$$\infty = \sum_{n=1}^{\infty} \underbrace{\mathbf{P}(X_1 > a_n)}_{=\mathbf{P}(X_n > a_n)} = \sum_{n=1}^{\infty} \mathbf{P}(X_n > a_n)$$

Then by Borel zero-one law, we have $\mathbf{P}(\{X_n > a_n\} \text{ i.o.}) = 1$.

The other direction, still use contradiction. Suppose $\mathbf{P}(A_n \text{ i.o.}) = 1$, but we have $\sum_n \mathbf{P}(X_1 > a_n) < \infty$. By IID, we have

$$\infty > \sum_{n=1}^{\infty} \underbrace{\mathbf{P}(X_1 > a_n)}_{=\mathbf{P}(X_n > a_n)} = \sum_{n=1}^{\infty} \mathbf{P}(X_n > a_n)$$

Then by Borel zero-one law, we have $\mathbf{P}(\{X_n > a_n\} \text{ i.o.}) = 0$, which contradicts the statement $\mathbf{P}(A_n \text{ i.o.}) = 1$.

11 Midterm 1

Problem 1. (4 points)

For any $\mathscr{C} \subset \mathcal{P}(\Omega)$, show that $\sigma(\mathscr{C}) = \sigma(\mathscr{A}(\mathscr{C}))$. Here $\mathscr{A}(\mathscr{C})$ is the algebra generated by $\mathscr{C}$.
Solution: We show

1) $\sigma(\mathscr{C}) \subseteq \sigma(\mathscr{A}(\mathscr{C}))$

2) $\sigma(\mathscr{C}) \supseteq \sigma(\mathscr{A}(\mathscr{C}))$

(1)

This is easy direction. Because $\mathscr{C} \subseteq \mathscr{A}(\mathscr{C})$, hence $\mathscr{C} \subseteq \sigma(\mathscr{A}(\mathscr{C}))$. By the definition of the smallest σ-algebra, we have $\sigma(\mathscr{C}) \subseteq \sigma(\mathscr{A}(\mathscr{C}))$

(2)

We want to show $\mathscr{A}(\mathscr{C}) \subseteq \sigma(\mathscr{C})$. Clearly algebra generated by $\mathscr{C}$ is contained in σ-algebra generated by $\mathscr{C}$. Because any sets in $\mathscr{A}(\mathscr{C})$ is constructed by taking set complement, set difference, finite union, finite intersection of sets in $\mathscr{C}$. On the other hand, $\sigma(\mathscr{C})$ is constructed by taking set complement, set difference, countable union, and countable intersection of sets in $\mathscr{C}$. So clearly $\mathscr{A}(\mathscr{C}) \subseteq \sigma(\mathscr{C})$. When taking countable unions of sets in $\mathscr{A}(\mathscr{C})$, we can take non-empty $A_1, \ldots, A_n \in \mathscr{C}$ and $\varnothing = A_{n+1} = A_{n+2} = \ldots$. Then under this construction, $\cup_{k=1}^{n} A_k = \cup_{k=1}^{\infty} A_k \in \sigma(\mathscr{C})$. This shows any set in $\mathscr{A}(\mathscr{C})$ will also be in $\sigma(\mathscr{C})$. Hence, we complete the proof.

Problem 2. (4 points)

Let $A \in \sigma(\mathscr{C})$, where $\mathscr{C}$ is an uncountable collection of subsets of Ω. Then there exists a countable subset $\mathscr{C}_0$ of $\mathscr{C}$ such that $A \in \sigma(\mathscr{C}_0)$.
Solution: The statement says for any $A \in \sigma(\mathscr{C})$, there exists a countable sub-collection $\mathscr{C}_0 \subset \mathscr{C}$ such that $A \in \sigma(\mathscr{C}_0)$. If this statement is true, then this σ-algebra is going to be union of all σ-algebras generated by some countable sub-collection $\mathcal{C}_0$, i.e.

$$\sigma(\mathscr{C}) = \bigcup_{\mathscr{C}_0 \subset \mathscr{C} \text{ countable}} \sigma(\mathscr{C}_0)$$

Define

$$\mathscr{H} := \{\mathscr{C}_0 \subsetneq \mathscr{C} : \mathscr{C}_0 \text{ is countable}\}$$

and denote

$$\mathscr{S} := \bigcup_{\mathscr{C}_0 \subset \mathscr{C} \text{ countable}} \sigma(\mathscr{C}_0) = \bigcup_{\mathscr{C}_0 \in \mathscr{H}} \sigma(\mathscr{C}_0)$$

We want to show $\mathscr{S} = \sigma(\mathscr{C})$. We will do this in the following steps:

1. Show $\mathscr{S} \subseteq \sigma(\mathscr{C})$
2. Show $\mathscr{C} \subset \mathscr{S}$.

3. Show $\mathscr{S}$ is a σ-algebra.

If we can show Step 2 and 3, then we can conclude $\sigma(\mathscr{C}) \subseteq \mathscr{S}$ by the definition of the smallest σ-algebra. Then it completes the proof.

(1)

It is clear that $\mathscr{S} \subseteq \sigma(\mathscr{C})$.

- Because $\mathscr{C}_0 \subsetneq \mathscr{C}$, hence $\mathscr{C}_0 \subset \sigma(\mathscr{C})$.
- Hence, by the definition of the smallest σ-algebra generated by $\mathscr{C}_0$, we have $\sigma(\mathscr{C}_0) \subset \sigma(\mathscr{C})$.
- Since this is true for all $\mathscr{C}_0 \in \mathscr{H}$, hence $\mathscr{S} := \cup_{\mathscr{C}_0 \in \mathscr{H}} \sigma(\mathscr{C}_0) \subseteq \sigma(\mathscr{C})$.

(2)

To show $\mathscr{C} \subset \mathscr{S}$. Let $A \in \mathscr{C}$. Then there will be a countable sub-collection of sets $\mathscr{C}_0 \subsetneq \mathscr{C}$ such that $A \in \mathscr{C}_0$. (For example, $\mathscr{C}_0 = \{A\}$ which only contains a single element). Thus, $A \in \sigma(\mathscr{C}_0)$ for some $\mathscr{C}_0 \in \mathscr{H}$. Hence, $A \in \mathscr{S} = \cup_{\mathscr{C}_0 \in \mathscr{H}} \sigma(\mathscr{C}_0)$. This completes the proof of $\mathscr{C} \subset \mathscr{S}$.

(3)

Because $\sigma(\mathscr{C}_0)$ is a σ-algebra for all $\mathscr{C}_0 \in \mathscr{H}$, hence $\Omega \in \sigma(\mathscr{C}_0)$. Hence, $\Omega \in \cup_{\mathscr{C}_0 \in \mathscr{H}} \sigma(\mathscr{C}_0) = \mathscr{S}$.

Let $A \in \mathscr{S}$. Then there exists at least one $\mathscr{C}_0 \in \mathscr{H}$ such that $A \in \sigma(\mathscr{C}_0)$. Because $\sigma(\mathscr{C}_0)$ is a σ-algebra, which means it is closed under complement. Hence $A^c \in \sigma(\mathscr{C}_0)$. Hence, $A^c \in \mathscr{S}$.

Let $A_1, \ldots, A_n, \ldots \in \mathscr{S}$. Then for each A_n, there exists $\mathscr{C}_n \in \mathscr{H}$ such that $A_n \in \sigma(\mathscr{C}_n)$. Define a set

$$\mathscr{C}_0 = \bigcup_{n=1}^{\infty} \mathscr{C}_n$$

- Because $\mathscr{C}_0$ is a countable union of countable sets, hence $\mathscr{C}_0$ is also countable. Hence, $\sigma(\mathscr{C}_0) \subset \mathscr{S}$ by construction of $\mathscr{S}$.
- And because $\mathscr{C}_n \subset \mathscr{C}_0$ for all $n \in \mathbb{Z}_{\geqslant 1}$, we have $\mathscr{C}_n \subset \sigma(\mathscr{C}_0)$ for all $n \in \mathbb{Z}_{\geqslant 1}$.
- Hence, by the definition of the smallest σ-algebra, we have $\sigma(\mathscr{C}_n) \subset \sigma(\mathscr{C}_0)$ for all $n \in \mathbb{Z}_{\geqslant 1}$.
- Hence, we conclude $A_n \in \sigma(\mathscr{C}_n) \subset \sigma(\mathscr{C}_0)$ for all $n \in \mathbb{Z}_{\geqslant 1}$.
- Because $\sigma(\mathscr{C}_0)$ is a σ-algebra, we conclude $\cup_{n=1}^{\infty} A_n \in \sigma(\mathscr{C}_0) \subset \mathscr{S}$.

This completes the proof of $\mathscr{S}$ being a σ-algebra.

Problem 3. (4 points)

Let $F(x)$ be a distribution function. Set $\mathscr{S} = \{(a,b] : -\infty \leqslant a \leqslant b \leqslant +\infty\}$, with the convention that $(a,\infty] = (a,\infty)$. Define a set function $\mathbf{P}$ on $\mathscr{S}$ by $\mathbf{P}((a,b]) = F(b) - F(a)$. Show that $\mathbf{P}$ is a probability measure on $\mathscr{S}$.

Solution:

(1) Show $\mathbf{P}(\Omega) = 1$

The whole set in this case is $\Omega = [-\infty, \infty]$. Then

$$\mathbf{P}([-\infty,\infty]) = F(\infty) - F(-\infty) = 1 - 0 = 1$$

(2) Show $\mathbf{P}(A) \geqslant 0$ for all $A \in \mathscr{S}$

Also because a distribution function is non-decreasing, hence for all $a \leqslant b$, we will have $F(a) \leqslant F(b)$. Thus, for any $(a,b] \subset \Omega$, we will have

$$\mathbf{P}((a,b]) = F(b) - F(a) \geqslant 0$$

Note that we can evaluate $F(x)$ at $x = a$ because distribution function is right-continuous.

(3) Show countable additivity

(3.1) Step 1: Show finite additivity of $\mathbf{P}$

Let $(a_n, b_n] \subset \Omega$ where $n = 1, 2, \dots, N$ be pairwise disjoint intervals such that $\cup_{n=1}^{N}(a_n, b_n] \in \mathscr{S}$. Then the only possibility that can let their union sit in $\mathscr{S}$ is we have $b_n = a_{n+1}$ for all $n \geqslant 1$. Let's consider the finite-additivity first. See that $\{\cup_{n=1}^{N}(a_n, b_n]\}_{N \geqslant 1}$ is an increasing sequence of N, and $\inf_n a_n = a_1$ and $\sup_n b_n = b_N$. Thus,

$$\mathbf{P}\left(\bigcup_{n=1}^{N}(a_n, b_n]\right) = F\left(\sup_n b_n\right) - F\left(\inf_n a_n\right) = F(b_N) - F(a_1)$$

Do the telescoping, we have

$$\begin{aligned}
\mathbf{P}\left(\bigcup_{n=1}^{N}(a_n, b_n]\right) &= F\left(\sup_n b_n\right) - F\left(\inf_n a_n\right) = F(b_N) - F(a_1) \\
&= F(b_N) - \underbrace{F(b_1)}_{=F(a_2)} + F(b_1) - F(a_1) \\
&= F(b_N) - F(a_N) + F(b_{N-1}) - F(a_{N-1}) + \dots + F(b_1) - F(a_1) \\
&= \sum_{n=1}^{N}(F(b_n) - F(a_n)) \\
&= \sum_{n=1}^{N}\mathbf{P}((a_n, b_n])
\end{aligned}$$

Thus, this set function $\mathbf{P}$ satisfies finite additivity.

(3.2) Step 2: Show countable additivity

Let $(a_n, b_n] \subset \Omega$ where $n = 1, 2, \ldots$ be pairwise disjoint intervals such that $\cup_{n=1}^{\infty}(a_n, b_n] \in \mathscr{S}$. WLOG, let's order them as $-\infty \leqslant a_1 \leqslant b_1 \leqslant a_2 \leqslant b_2 \leqslant \ldots \leqslant +\infty$. More precisely, the union will be of the form

$$\bigcup_{n=1}^{\infty}(a_n, b_n] = (a, b]$$

where $a = \inf_n a_n$ and $b = \sup_n b_n$.

Want to show

$$F(b) - F(a) = \mathbf{P}((a, b]) = \sum_{i=1}^{\infty}\mathbf{P}((a_i, b_i]) = \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$$

The problem of doing a countable summation of positive numbers, things can really go wrong. For example, this summation may goes to infinity. You don't know if this countable sum will converge or not.

(3.2.1) Step 1: show $F(b) - F(a) \leqslant \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$

What we will do is to fix a small $\epsilon > 0$. And we will make the interval $(a, b]$ a bit smaller by looking at the interval $[a + \epsilon, b]$. In addition, we will make the intervals $(a_i, b_i]$ slightly bigger by looking at $(a_i, b_i + \epsilon_i)$ where $\epsilon_i > 0$ is a small positive number such that

$$\mathbf{P}((a_i, b_i + \epsilon_i]\backslash(a_i, b_i]) = \mathbf{P}((a_i, b_i + \epsilon_i]) - \mathbf{P}((a_i, b_i]) = F(b_i + \epsilon_i) - F(b_i) \leqslant \epsilon\frac{1}{2^i}$$

Such interval $(a_i, b_i + \epsilon_i)$ exists because of the right-continuity of F. See that if we let $\epsilon_i \downarrow 0$. Then the CDF will be

$$\lim_{\epsilon_i \downarrow 0} F(b_i + \epsilon_i) = F(b_i+) = F(b_i)$$

Thus, taking limit of probability difference will become

$$\lim_{\epsilon_i \downarrow 0}(\mathbf{P}((a_i, b_i + \epsilon_i]) - \mathbf{P}((a_i, b_i])) = \lim_{\epsilon_i \downarrow 0}(F(b_i + \epsilon_i) - F(b_i)) = F(b_i) - F(b_i) = 0$$

Hence, we can choose some $\epsilon_i > 0$ such that the desired property holds. Then we have

$$[a + \epsilon, b] \subset (a, b] = \sum_{i \geqslant 1}(a_i, b_i] \subset \bigcup_{i=1}^{\infty}\Big(a_i, b_i + \epsilon_i\Big)$$

By (quasi-)compactness of the interval $[a + \epsilon, b]$, every open cover has a finite subcover. Hence, there exists N such that

$$[a + \epsilon, b] \subset \bigcup_{i=1}^{N}\Big(a_i, b_i + \epsilon_i\Big)$$

Hence, we have

$$(a + \epsilon, b] \subset [a + \epsilon, b] \subset \bigcup_{i=1}^{N}\Big(a_i, b_i + \epsilon_i\Big) \subset \bigcup_{i=1}^{N}\Big(a_i, b_i + \epsilon_i\Big]$$

So by monotonicity of probability measure and finite additivity of $\mathbf{P}$,

$$
\begin{aligned}
F(b) - F(a+\epsilon) = \mathbf{P}((a+\epsilon, b]) \leqslant \mathbf{P}\Big(\bigcup_{i=1}^{N}(a_i, b_i+\epsilon_i]\Big) &= \sum_{i=1}^{N}\mathbf{P}\Big((a_i, b_i+\epsilon_i]\Big) \\
&= \sum_{i=1}^{N}\Big(F(b_i+\epsilon_i) - F(a_i)\Big) \\
&= \sum_{i=1}^{N}\Big(\underbrace{F(b_i+\epsilon_i) - F(b_i)}_{\leqslant \frac{\epsilon}{2^i}} + F(b_i) - F(a_i)\Big) \\
&\leqslant \sum_{i=1}^{N}(F(b_i) - F(a_i)) + \epsilon\sum_{i=1}^{N}\frac{1}{2^i} \\
&\leqslant \sum_{i=1}^{\infty}(F(b_i) - F(a_i)) + \epsilon\sum_{i=1}^{\infty}\frac{1}{2^i} \\
&= \sum_{i=1}^{\infty}(F(b_i) - F(a_i)) + \epsilon
\end{aligned}
$$

Since ϵ is arbitrary, we send ϵ to 0, we get

$$F(b) - F(a) \leqslant \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$$

where in the last step we use right-continuity of F, where $\lim_{\epsilon\to 0} F(a+\epsilon) = F(a)$.

(3.2.2) Step 2: show $F(b) - F(a) \geqslant \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$

Next we show

$$F(b) - F(a) \geqslant \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$$

Note

$$(a, b] = \sum_{i=1}^{\infty}(a_i, b_i] \supset \sum_{i=1}^{N}(a_i, b_i], \qquad \text{for any } N \in \mathbb{Z}_{\geqslant 1}$$

Then by the monotonicity of probability measure, we have

$$F(b) - F(a) = \mathbf{P}((a, b]) \geqslant \mathbf{P}\Big(\sum_{i=1}^{N}(a_i, b_i]\Big) = \sum_{i=1}^{N}(F(b_i) - F(a_i))$$

Send N to infinity, we have

$$F(b) - F(a) \geqslant \sum_{i=1}^{\infty}(F(b_i) - F(a_i))$$

Problem 4. (4 points)

Prove the "Inclusion-Exclusion" formula (Formula (2.2) on Page 30 of the textbook).
Recall: The inclusion-exclusion formula is of the form

$$\mathbf{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n}\mathbf{P}(A_k) - \sum_{1\leqslant i<j\leqslant n}\mathbf{P}(A_i\cap A_j) + \sum_{1\leqslant i<j<k\leqslant n}\mathbf{P}(A_i\cap A_j\cap A_k) - \dots$$
$$+ (-1)^{n-1}\sum_{1\leqslant i_1<\ldots<i_n\leqslant n}\mathbf{P}(A_{i_1}\cap\ldots\cap A_{i_n})$$

For example, in the matching problem, where $\mathbf{P}(A_{i_1}\cap\cdots\cap A_{i_k}) = \frac{(n-k)!}{n!}$, this evaluates to

$$\binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} - \ldots + (-1)^{n-1}\frac{1}{n!}$$

Solution: Use proof by induction.

Base case, $n = 2$.

Because $A_1\cup A_2 = A_1\backslash A_2 + A_1\cap A_2 + A_2\backslash A_1$, then finite-additivity of probability measure,

$$\begin{aligned}\mathbf{P}(A_1\cup A_2) &= \mathbf{P}(A_1\backslash A_2) + \mathbf{P}(A_1\cap A_2) + \mathbf{P}(A_2\backslash A_1)\\ &= \mathbf{P}(A_1\backslash A_2) + \mathbf{P}(A_1\cap A_2) + \mathbf{P}(A_2\backslash A_1) + \mathbf{P}(A_1\cap A_2) - \mathbf{P}(A_1\cap A_2)\\ &= (\mathbf{P}(A_1\backslash A_2) + \mathbf{P}(A_1\cap A_2)) + (\mathbf{P}(A_2\backslash A_1) + \mathbf{P}(A_1\cap A_2)) - \mathbf{P}(A_1\cap A_2)\\ &= \mathbf{P}(A_1) + \mathbf{P}(A_2) - \mathbf{P}(A_1\cap A_2)\end{aligned}$$

which completes the proof of base case.

Induction case

Suppose the formula is true for all $k\leqslant n$ case, we want to verify it is also true for $k = n+1$ case. See that by apply $n = 2$ case to the following, we have

$$\mathbf{P}\left(\bigcup_{k=1}^{n+1} A_k\right) = \mathbf{P}\left(\bigcup_{k=1}^{n} A_k\cup A_{n+1}\right) = \mathbf{P}\left(\bigcup_{k=1}^{n} A_k\right) + \mathbf{P}(A_{n+1}) - \mathbf{P}\left(\left(\bigcup_{k=1}^{n} A_k\right)\cap A_{n+1}\right)$$

See that by applying distributive law to the sets in the last term, we get

$$\mathbf{P}\left(\left(\bigcup_{k=1}^{n} A_k\right)\cap A_{n+1}\right) = \mathbf{P}\left(\bigcup_{k=1}^{n}(A_k\cap A_{n+1})\right)$$

Applying the induction hypothesis to $\mathbf{P}(\cup_{k=1}^{n}A_k)$ and $\mathbf{P}(\cup_{k=1}^{n}(A_k\cap A_{n+1}))$. We get

$$\mathbf{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n}\mathbf{P}(A_k) - \sum_{1\leqslant i<j\leqslant n}\mathbf{P}(A_iA_j) + \sum_{1\leqslant i<j<k\leqslant n}\mathbf{P}(A_iA_jA_k) - \dots$$
$$+ (-1)^{n-1}\,\mathbf{P}(A_1\ldots A_n)$$

and

$$\begin{aligned}\mathbf{P}\left(\bigcup_{k=1}^{n}(A_k \cap A_{n+1})\right) = &\sum_{k=1}^{n}\mathbf{P}(A_kA_{n+1}) - \sum_{1\leqslant i<j\leqslant n}\underbrace{\mathbf{P}((A_iA_{n+1})(A_jA_{n+1}))}_{=\mathbf{P}(A_iA_jA_{n+1})} \\ &+ \sum_{1\leqslant i<j<k\leqslant n}\underbrace{\mathbf{P}((A_iA_{n+1})(A_jA_{n+1})(A_kA_{n+1}))}_{=\mathbf{P}(A_iA_jA_kA_{n+1})} - \dots \\ &+ (-1)^{n-1}\underbrace{\mathbf{P}((A_1A_{n+1})\dots(A_nA_{n+1}))}_{=\mathbf{P}(A_1\dots A_nA_{n+1})}\end{aligned}$$

Note that

$$\begin{aligned}\mathbf{P}((A_iA_{n+1})(A_jA_{n+1})) &= \mathbf{P}(A_iA_jA_{n+1}) \\ \mathbf{P}((A_iA_{n+1})(A_jA_{n+1})(A_kA_{n+1})) &= \mathbf{P}(A_iA_jA_kA_{n+1}) \\ &\vdots \\ \mathbf{P}((A_1A_{n+1})\dots(A_nA_{n+1})) &= \mathbf{P}(A_1\dots A_nA_{n+1})\end{aligned}$$

Plug all of these back into the original equation, we get

$$\begin{aligned}
\mathbf{P}\left(\bigcup_{k=1}^{n+1} A_k\right) &= \mathbf{P}\left(\bigcup_{k=1}^{n} A_k\right) + \mathbf{P}(A_{n+1}) - \mathbf{P}\left(\left(\bigcup_{k=1}^{n} A_k\right) \cap A_{n+1}\right) \\
&= \left(\begin{aligned}\sum_{k=1}^{n} \mathbf{P}(A_k) - \sum_{1\leqslant i<j\leqslant n} \mathbf{P}(A_iA_j) + \sum_{1\leqslant i<j<k\leqslant n} \mathbf{P}(A_iA_jA_k) - \ldots \\ + (-1)^{n-1}\,\mathbf{P}(A_1\ldots A_n)\end{aligned}\right) \\
&\qquad + \mathbf{P}(A_{n+1}) \\
&\quad - \left(\begin{aligned}&\sum_{k=1}^{n} \mathbf{P}(A_kA_{n+1}) - \sum_{1\leqslant i<j\leqslant n} \underbrace{\mathbf{P}((A_iA_{n+1})(A_jA_{n+1}))}_{=\mathbf{P}(A_iA_jA_{n+1})} \\
&\quad + \sum_{1\leqslant i<j<k\leqslant n} \underbrace{\mathbf{P}((A_iA_{n+1})(A_jA_{n+1})(A_kA_{n+1}))}_{=\mathbf{P}(A_iA_jA_kA_{n+1})} - \ldots \\
&\quad + (-1)^{n-2} \sum_{1\leqslant i_1<\ldots<i_{n-1}\leqslant n} \mathbf{P}(A_{i_1}\cap\ldots\cap A_{i_{n-1}}\cap A_{n+1}) \\
&\qquad + (-1)^{n-1}\,\underbrace{\mathbf{P}((A_1A_{n+1})\ldots(A_nA_{n+1}))}_{=\mathbf{P}(A_1\ldots A_nA_{n+1})}\end{aligned}\right) \\
&= \sum_{k=1}^{n+1} \mathbf{P}(A_k) - \underbrace{\left(\sum_{1\leqslant i<j\leqslant n} \mathbf{P}(A_iA_j) + \sum_{k=1}^{n} \mathbf{P}(A_kA_{n+1})\right)}_{=\sum_{1\leqslant i<j\leqslant n+1}\mathbf{P}(A_iA_j)} \\
&\quad + \underbrace{\left(\sum_{1\leqslant i<j<k\leqslant n} \mathbf{P}(A_iA_jA_k) + \sum_{1\leqslant i<j\leqslant n} \mathbf{P}(A_iA_jA_{n+1})\right)}_{=\sum_{1\leqslant i<j<k\leqslant n+1}\mathbf{P}(A_iA_jA_k)} \\
&\quad - \ldots + (-1)^{n-1}\underbrace{\left(\mathbf{P}(A_1\ldots A_n) + \sum_{1\leqslant i_1<\ldots<i_{n-1}\leqslant n} \mathbf{P}(A_{i_1}\ldots A_{i_{n-1}}A_{n+1})\right)}_{=\sum_{1\leqslant i_1<\ldots<i_n\leqslant n+1}\mathbf{P}(A_{i_1}\ldots A_{i_n})} \\
&\qquad + (-1)^{n}\,\mathbf{P}(A_1\ldots A_nA_{n+1}) \\
&= \sum_{k=1}^{n+1} \mathbf{P}(A_k) - \sum_{1\leqslant i<j\leqslant n+1} \mathbf{P}(A_iA_j) + \sum_{1\leqslant i<j<k\leqslant n+1} \mathbf{P}(A_iA_jA_k) - \ldots \\
&\qquad + (-1)^{n}\,\mathbf{P}(A_1\ldots A_{n+1})
\end{aligned}$$

Hence, we completes the proof by induction.

Problem 5 (4 points)

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space. A set $A \in \mathscr{F}$ of positive measure is called an atom of $\mathbf{P}$ if $\mathbf{P}(B) = 0$ or $\mathbf{P}(B) = \mathbf{P}(A)$ for every subset $B \in \mathscr{F}$ of A. A measure is called non-atomic if it does not have atoms. Suppose that $\mathbf{P}$ is a non-atomic measure

in this problem. Show the following is true: Fix any set $A \in \mathscr{F}$ of positive measure, i.e. $\mathbf{P}(A) > 0$. For arbitrary small $\epsilon > 0$, there is a subset B of A such that $0 < \mathbf{P}(B) < \epsilon$.
Solution: Fix an arbitrary small $\epsilon > 0$. Fix a set $A \in \mathscr{F}$ with $\mathbf{P}(A) > 0$. Because $\mathbf{P}$ is non-atomic, there exists a $B \in \mathscr{F}$ such that $B \subset A$ and $0 < \mathbf{P}(B) < \mathbf{P}(A)$. Then inductively, let's do the same thing on B.

For convenience, let's denote $A = A_1$. Then because $\mathbf{P}$ is non-atomic, there exists $A_2 \in \mathscr{F}$ such that $A_2 \subset A_1$ and $0 < \mathbf{P}(A_2) < \mathbf{P}(A_1)$. Keep going, there exists $A_3 \in \mathscr{F}$ such that $A_3 \subset A_2$ and $0 < \mathbf{P}(A_3) < \mathbf{P}(A_2)$. And so on.

Because the inequality is $<$ not $\leqslant$. There will be some $N \geqslant 1$ such that $A_N \subset A_{N-1}$ and $0 < \mathbf{P}(A_N) < \epsilon$. We define this $B = A_N$.

Remark

Last answer is wrong because although $\mathbf{P}(A_n)$ might be decreasing, its limit might not be zero. The counter-example is

$$\mathbf{P}(A_n) = \frac{1}{2} + \frac{1}{n}$$

Refinement: Use prove by contradiction. Suppose

$$\inf_{B \subset A, \mathbf{P}(B) > 0} \mathbf{P}(B) = \epsilon$$

for some $\epsilon > 0$.

Denote $B_0 = A$ and $\epsilon_0 = \epsilon$. Now define

- $\epsilon_1 = \inf_{B \subset B_0, \mathbf{P}(B) > 0} \mathbf{P}(B)$. Since $B_0 = A$, this is the same infimum as the one defining ϵ_0, so $\epsilon_1 = \epsilon_0$.
- Pick $B_1 \subset B_0$ such that $\epsilon_1 \leqslant \mathbf{P}(B_1) < \epsilon_0 + 1$.

Next, we define inductively that

1) Define $\epsilon_n = \inf_{B \subset B_{n-1}, \mathbf{P}(B) > 0} \mathbf{P}(B)$. Then clearly by construction $\epsilon_n \geqslant \epsilon_{n-1} \geqslant \epsilon$.
2) Pick $B_n \subseteq B_{n-1}$ such that $\epsilon_n \leqslant \mathbf{P}(B_n) < \epsilon_{n-1} + \frac{1}{n}$.

We have that the two sequences produced by this construction are monotone in opposite directions.

- (ϵ_n) is non-decreasing. Indeed $B_n \subseteq B_{n-1}$, so the infimum defining ϵ_{n+1} ranges over a sub-collection of the one defining ϵ_n, and an infimum over a smaller collection is at least as large. Moreover $\mathbf{P}(B_{n-1}) \geqslant \epsilon_{n-1} \geqslant \epsilon > 0$, so B_{n-1} is itself admissible in the infimum defining ϵ_n and $\epsilon_n \leqslant \mathbf{P}(B_{n-1}) \leqslant 1$. A non-decreasing sequence bounded above converges, say $\epsilon_n \uparrow \epsilon_\infty \in [\epsilon, 1]$.
- $(\mathbf{P}(B_n))$ is non-increasing, since $B_n \subseteq B_{n-1}$, and bounded below by $\epsilon_n \geqslant \epsilon > 0$.

The only consequence of 2) used below is the two-sided estimate

$$\epsilon_n \leqslant \mathbf{P}(B_n) < \epsilon_{n-1} + \frac{1}{n} \leqslant \epsilon_n + \frac{1}{n}$$

the last step being the monotonicity just recorded. (A set B_n meeting this weaker requirement always exists, straight from the definition of ϵ_n as an infimum.) Squeezing and using $\epsilon_n \uparrow \epsilon_\infty$,

$$\lim_{n\to\infty} \mathbf{P}(B_n) = \epsilon_\infty$$

Now put

$$B := \bigcap_{n\geqslant 0} B_n \in \mathscr{F}$$

The B_n decrease, so continuity from above of $\mathbf{P}$ gives

$$\mathbf{P}(B) = \lim_{n\to\infty} \mathbf{P}(B_n) = \epsilon_\infty \geqslant \epsilon > 0$$

and $B \subseteq B_0 = A$. Since $\mathbf{P}$ is non-atomic and $\mathbf{P}(B) > 0$, the set B is not an atom, so there is a $C \in \mathscr{F}$ with $C \subseteq B$ and

$$0 < \mathbf{P}(C) < \mathbf{P}(B) = \epsilon_\infty$$

On the other hand $C \subseteq B \subseteq B_{n-1}$ and $\mathbf{P}(C) > 0$, so C is admissible in the infimum defining ϵ_n, i.e.

$$\mathbf{P}(C) \geqslant \epsilon_n \qquad \text{for every } n \geqslant 1$$

Letting $n \to \infty$ gives $\mathbf{P}(C) \geqslant \epsilon_\infty$, contradicting $\mathbf{P}(C) < \epsilon_\infty$.

Hence the assumption that the infimum is positive is untenable, and

$$\inf_{B\subseteq A, \mathbf{P}(B)>0} \mathbf{P}(B) = 0$$

The collection over which this infimum is taken is non-empty, because A itself belongs to it. So by the definition of an infimum, for every $\delta > 0$ there is a $B \in \mathscr{F}$ with $B \subseteq A$ and

$$0 < \mathbf{P}(B) < \delta$$

which is exactly what the problem asks for.

12 Midterm 2

Problem 1. (5 points)

Let $(\Omega, \mathscr{B})$ be a measurable space and X a r.v. taking values in $\mathbb{R}$. Let $\sigma(X)$ be the σ-field generated by X and $\mathscr{B}(\mathbb{R})$ the Borel σ-field on $\mathbb{R}$.

Problem (1.1)

Show that if $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then $Z = f \circ X : \Omega \to \mathbb{R}$ is $\sigma(X)/\mathscr{B}(\mathbb{R})$ measurable.

Solution: If $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, it means $f^{-1}(\mathscr{B}(\mathbb{R})) \subseteq \mathscr{B}(\mathbb{R})$.

Since X is a random variable. So $\sigma(X) = X^{-1}(\mathscr{B}(\mathbb{R})) \subseteq \mathscr{B}$.

Because $f^{-1}(\mathscr{B}(\mathbb{R})) \subseteq \mathscr{B}(\mathbb{R})$, thus

$$Z^{-1}(\mathscr{B}(\mathbb{R})) = (f \circ X)^{-1}(\mathscr{B}(\mathbb{R})) = X^{-1}(f^{-1}(\mathscr{B}(\mathbb{R}))) \subseteq X^{-1}(\mathscr{B}(\mathbb{R})) = \sigma(X) \subseteq \mathscr{B}$$

Hence, we do have $Z = f \circ X$ is $\sigma(X)/\mathscr{B}(\mathbb{R})$ measurable.

Problem (1.2)

Show that, conversely, for any $Z : \Omega \to \mathbb{R}$ which is $\sigma(X)/\mathscr{B}(\mathbb{R})$ measurable, there exists a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $Z = f \circ X$. (Hint: start with indicator functions and approximate an $\sigma(X)/\mathscr{B}(\mathbb{R})$ function by simple functions.)

Solution:

Step 1: Let $Z = \mathbf{1}_A$ for some $A \in \sigma(X)$

Because Z is $\sigma(X)/\mathscr{B}(\mathbb{R})$-measurable, we have $Z^{-1}(\mathscr{B}(\mathbb{R})) \subseteq \sigma(X) = X^{-1}(\mathscr{B}(\mathbb{R}))$.

Because $A \in \sigma(X)$, it means $A = X^{-1}(B)$ for some $B \in \mathscr{B}(\mathbb{R})$. Hence,

$$Z = \mathbf{1}_A = \mathbf{1}_{X^{-1}(B)} = \mathbf{1}_B \circ X$$

Hence, we may choose $f = \mathbf{1}_B$. Because $B \in \mathscr{B}(\mathbb{R})$, so f is clearly Borel measurable.

Step 2: Let $Z = \sum_{i=1}^n a_i \mathbf{1}_{A_i} \in \mathcal{E}$ be a simple function

The result will just extend by linearity. Because each $A_i = X^{-1}(B_i)$ for some $B_i \in \mathscr{B}(\mathbb{R})$.

Thus,

$$Z = \sum_{i=1}^n a_i \mathbf{1}_{A_i} = \sum_{i=1}^n a_i \mathbf{1}_{X^{-1}(B_i)} = \sum_{i=1}^n a_i \mathbf{1}_{B_i} \circ X = \left(\sum_{i=1}^n a_i \mathbf{1}_{B_i} \right) \circ X$$

Hence, we may choose $f = \sum_{i=1}^n a_i \mathbf{1}_{B_i}$.

Step 3: Let $Z \geqslant 0$ be a non-negative $\sigma(X)/\mathscr{B}(\mathbb{R})$-measurable function

For $Z \geqslant 0$ and $\sigma(X)/\mathscr{B}(\mathbb{R})$-measurable, we can pick an approximating sequence $0 \leqslant Z_n \uparrow Z$. Note that in this case

$$Z = \lim_{n \to \infty} Z_n = \sup_n Z_n$$

From Step (2), we know for each $Z_n \in \mathcal{E}$, there is a Borel measurable $f_n \in \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$ such that $Z_n = f_n \circ X$.

Now the issue is we don't know

- if $\lim_{n\to\infty} f_n$ exists or no (from the construction, there is no monotone behaviour of f_n, so we don't know if f_n converge or not).

But $\sup_n f_n$ will exist because following from construction from Step (2), we see that each f_n is a simple function, so each one of them is finite. Hence, it makes sense to define $\sup_n f_n$. Hence, define

$$f := \sup_n f_n$$

This f is again Borel measurable. Observe that

$$Z = \sup_n Z_n = \sup_n (f_n \circ X) = \left(\sup_n f_n\right) \circ X = f \circ X$$

Step 4: General $\sigma(X)/\mathscr{B}(\mathbb{R})$-measurable RV Z

Separate Z using the trick

$$Z = Z^+ - Z^-$$

where $Z^+ = \max\{Z, 0\} \geqslant 0$ and $Z^- = -\min\{Z, 0\} \geqslant 0$. From Step (3), we know there are Borel measurable functions $f_1, f_2 \in \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$ such that

$$Z^+ = f_1 \circ X, \qquad Z^- = f_2 \circ X$$

Hence,

$$Z = Z^+ - Z^- = f_1 \circ X - f_2 \circ X = (f_1 - f_2) \circ X = f \circ X$$

where we define $f := f_1 - f_2$. This is clearly Borel measurable.

This completes the proof.

Problem 2 (5 points)

Let $(\Omega, \mathscr{B}, \mu)$ be a probability space, and $f \geqslant 0$ such that $\int_\Omega f d\mu = 1$. Define a set function ν on $(\Omega, \mathscr{B})$ by

$$\nu(B) = \int_B f \, d\mu, \qquad \forall\ B \in \mathscr{B}$$

Problem (2.1)

Show that ν is a probability measure.

Solution:

Show $\nu(\Omega) = 1$

See that by our assumption, we have

$$\nu(\Omega) = \int_\Omega f \, d\mu = 1$$

Show if $A \in \mathscr{B}$, then $\nu(A) \geqslant 0$.

Let $A \in \mathscr{B}$. Using the property $X \geqslant Y$ then $\mathbf{E}X \geqslant \mathbf{E}Y$, we have

$$\nu(A) = \int_A f\ d\mu = \int_\Omega \underbrace{\mathbf{1}_A f}_{\geqslant 0}\ d\mu \geqslant 0$$

Show σ-additivity

Let $A_1, \ldots, A_n, \ldots \in \mathscr{B}$ be disjoint and $A = \cup_{n=1}^\infty A_n \in \mathscr{B}$. Want to show $\nu(A) = \sum_{n=1}^\infty \nu(A_n)$.

$$\begin{aligned}
\nu(A) &= \int_A f\ d\mu = \int_\Omega \mathbf{1}_A f\ d\mu = \int_\Omega \mathbf{1}_{\cup_{n=1}^\infty A_n} f\ d\mu \\
&= \int_\Omega \sum_{n=1}^\infty \mathbf{1}_{A_n} f\ d\mu = \int_\Omega \lim_{N\to\infty} \sum_{n=1}^N \mathbf{1}_{A_n} f\ d\mu \\
&\overset{\text{MCT}}{=} \lim_{N\to\infty} \int_\Omega \sum_{n=1}^N \mathbf{1}_{A_n} f\ d\mu \\
&= \lim_{N\to\infty} \sum_{n=1}^N \int_\Omega \mathbf{1}_{A_n} f\ d\mu, \qquad \text{by linearity of integral} \\
&= \sum_{n=1}^\infty \int_{A_n} f\ d\mu = \sum_{n=1}^\infty \nu(A_n)
\end{aligned}$$

This completes the proof.

Problem (2.2)

Show that for any random variable $X \geqslant 0$, we have

$$\int X d\nu = \int X f d\mu$$

Hint: Start with indicator functions for X

Solution:

Step 1: Indicator functions

First, let's assume $X = \mathbf{1}_A$ for some $A \in \mathscr{B}$, and show above is true for indicator function. Simply observe that

$$\int_\Omega \mathbf{1}_A d\nu = \nu(A) = \int_A f d\mu = \int_\Omega \mathbf{1}_A f d\mu$$

Hence, the statement is true for indicator functions.

Step 2: Simple functions

Next, let's assume $X = \sum_{i=1}^n a_i \mathbf{1}_{A_i}$ where $a_i \in \mathbb{R}_{\geqslant 0}$ and A_i's is a partition of Ω. Then we have

$$\begin{aligned}
\int_\Omega X d\nu &= \int_\Omega \sum_{i=1}^n a_i \mathbf{1}_{A_i} d\nu = \sum_{i=1}^n a_i \int_\Omega \mathbf{1}_{A_i} d\nu \\
&= \sum_{i=1}^n a_i \int_\Omega \mathbf{1}_{A_i} f d\mu = \int_\Omega \sum_{i=1}^n a_i \mathbf{1}_{A_i} f d\mu = \int_\Omega \left(\sum_{i=1}^n a_i \mathbf{1}_{A_i}\right) f d\mu = \int_\Omega X f d\mu
\end{aligned}$$

where we applied linearity of integral in the process. Hence, we have done the simple function part.

Step 3: Non-negative function.

Denote $\mathcal{E}$ be the collection of simple functions. If $X \geqslant 0$, and $X_n \in \mathcal{E}$ (with $X_n \geqslant 0$) is an (increasing) approximating sequence of X, i.e. $X_n \uparrow X$ as $n \to \infty$.

Then

$$\begin{aligned}
\int_\Omega X d\nu &= \int_\Omega \lim_{n\to\infty} X_n d\nu \\
&= \lim_{n\to\infty} \int_\Omega X_n d\nu, \qquad \text{by MCT} \\
&= \lim_{n\to\infty} \int_\Omega X_n f\ d\mu, \qquad \text{using Step (2)} \\
&= \int_\Omega \lim_{n\to\infty} X_n f\ d\mu = \int_\Omega X f\ d\mu, \qquad \text{by MCT}
\end{aligned}$$

Here, we applied Monotone Convergence theorem twice. First time is easy. The second time is because, if $X_n \uparrow X$, then $X_n f \uparrow X f$. Hence, it is valid to apply MCT. Thus, we complete the proof.

Problem 3. (5 points)

Suppose X and Y are independent random variables. Show that for any measurable function $f, g : \mathbb{R} \to \mathbb{R}$ we have that $f(X)$ and $g(Y)$ are independent.

Solution: Here, the compositions of the maps are

$$(\Omega, \mathscr{B}, \mathbf{P}) \xrightarrow{X,Y} (\mathbb{R}, \mathscr{B}(\mathbb{R})) \xrightarrow{f,g} (\mathbb{R}, \mathscr{B}(\mathbb{R}))$$

Suppose X, Y are independent. That means $\sigma(X) = X^{-1}(\mathscr{B}(\mathbb{R}))$ and $\sigma(Y) = Y^{-1}(\mathscr{B}(\mathbb{R}))$ are independent. This means for all events $A \in \sigma(X)$ and $B \in \sigma(Y)$, A and B are independent.

Because $f, g \in \mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$ are measurable. Hence,

$$f^{-1}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}(\mathbb{R}), \qquad g^{-1}(\mathscr{B}(\mathbb{R})) \subset \mathscr{B}(\mathbb{R})$$

Thus,

$$\begin{aligned}
\sigma(f \circ X) = (f \circ X)^{-1}(\mathscr{B}(\mathbb{R})) = X^{-1}(f^{-1}(\mathscr{B}(\mathbb{R}))) \subset \sigma(X), \\
\sigma(g \circ Y) = (g \circ Y)^{-1}(\mathscr{B}(\mathbb{R})) = Y^{-1}(g^{-1}(\mathscr{B}(\mathbb{R}))) \subset \sigma(Y)
\end{aligned}$$

Thus, any $A \in \sigma(f \circ X)$ and any $B \in \sigma(g \circ Y)$ are independent. Thus, $\sigma(f \circ X)$ and $\sigma(g \circ Y)$ are independent, which means $f(X)$ and $g(Y)$ are independent.

Problem 4. (5 points)

Let $(\Omega, \mathscr{B}, \mathbf{P})$ be a probability space and $X \in L^1(\mathbb{R})$ a random variable. Show that for any fixed $\epsilon > 0$, there always exists a $\delta > 0$ such that for all $B \in \mathscr{B}$ with $\mathbf{P}(B) < \delta$, we have

$$\int_B |X| d\mathbf{P} < \epsilon$$

Solution: $X \in L^1(\mathbb{R})$ means $\int_\Omega |X(\omega)| d\,\mathbf{P}(\omega) < \infty$. Suppose for contradiction that there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exists a $B \in \mathscr{B}$ with $\mathbf{P}(B) < \delta$ but

$$\int_B |X| d\,\mathbf{P} \geqslant \epsilon$$

Because the above statement is for all $\delta > 0$, let's construct a sequence $\delta_n = \frac{1}{2^n}$ for $n \in \mathbb{Z}_{\geqslant 1}$. Then for each δ_n, there exists a $B_n \in \mathscr{B}$ with $\mathbf{P}(B_n) < \delta_n$ such that

$$\int_{B_n} |X| d\,\mathbf{P} \geqslant \epsilon$$

Then integrate $|X|$ over B_n

$$\begin{aligned}
\int_{B_n} |X| d\,\mathbf{P} &= \int_{B_n} |X| (\mathbf{1}_{\{|X| \leqslant n\}} + \mathbf{1}_{\{|X| > n\}}) d\,\mathbf{P} = \int_{B_n} |X| \mathbf{1}_{\{|X| \leqslant n\}} d\,\mathbf{P} + \int_{B_n} |X| \mathbf{1}_{\{|X| > n\}} d\,\mathbf{P} \\
&\leqslant n\,\mathbf{P}(B_n) + \int_{B_n} |X| \mathbf{1}_{\{|X| > n\}} d\,\mathbf{P} \\
&\leqslant n \frac{1}{2^n} + \int_\Omega |X| \mathbf{1}_{\{|X| > n\}} d\,\mathbf{P}
\end{aligned}$$

Because $\int_{B_n} |X| d\,\mathbf{P} \geqslant \epsilon$, hence we have

$$\epsilon \leqslant \int_{B_n} |X| d\,\mathbf{P} \leqslant n \frac{1}{2^n} + \int_\Omega |X| \mathbf{1}_{\{|X| > n\}} d\,\mathbf{P}$$

Now let $n \to \infty$. See that

1. $\frac{n}{2^n} \to 0$ as $n \to \infty$.
2. $|X| \mathbf{1}_{\{|X| > n\}} \downarrow 0$ almost surely as $n \to \infty$ because $X \in L^1(\mathbb{R})$.

To see that the second term tends to 0, define

$$g_n := |X| \mathbf{1}_{\{|X| > 1\}} - |X| \mathbf{1}_{\{|X| > n\}}, \qquad n \geqslant 1$$

Since $\mathbf{1}_{\{|X| > n\}} \leqslant \mathbf{1}_{\{|X| > 1\}}$ for $n \geqslant 1$, each $g_n \geqslant 0$; the g_n are non-decreasing in n, and by 2. above $g_n \uparrow |X| \mathbf{1}_{\{|X| > 1\}}$ almost surely. By the Monotone Convergence Theorem,

$$\mathbf{E}\, g_n \uparrow \mathbf{E}\big[|X| \mathbf{1}_{\{|X| > 1\}}\big] \leqslant \mathbf{E}|X| < \infty$$

and therefore

$$\int_\Omega |X| \mathbf{1}_{\{|X| > n\}} d\,\mathbf{P} = \mathbf{E}\big[|X| \mathbf{1}_{\{|X| > 1\}}\big] - \mathbf{E}\, g_n \longrightarrow 0 \qquad \text{as } n \to \infty$$

(the subtraction is legitimate because $\mathbf{E}[|X| \mathbf{1}_{\{|X| > 1\}}]$ is finite). Hence

$$\epsilon \leqslant \lim_{n \to \infty} \left(n \frac{1}{2^n} + \int_\Omega |X| \mathbf{1}_{\{|X| > n\}} d\,\mathbf{P} \right) = 0 + 0 = 0$$

This is a contradiction.

References

[Res99] Sidney I. Resnick. *A Probability Path.* Birkhäuser, Boston, 1999.